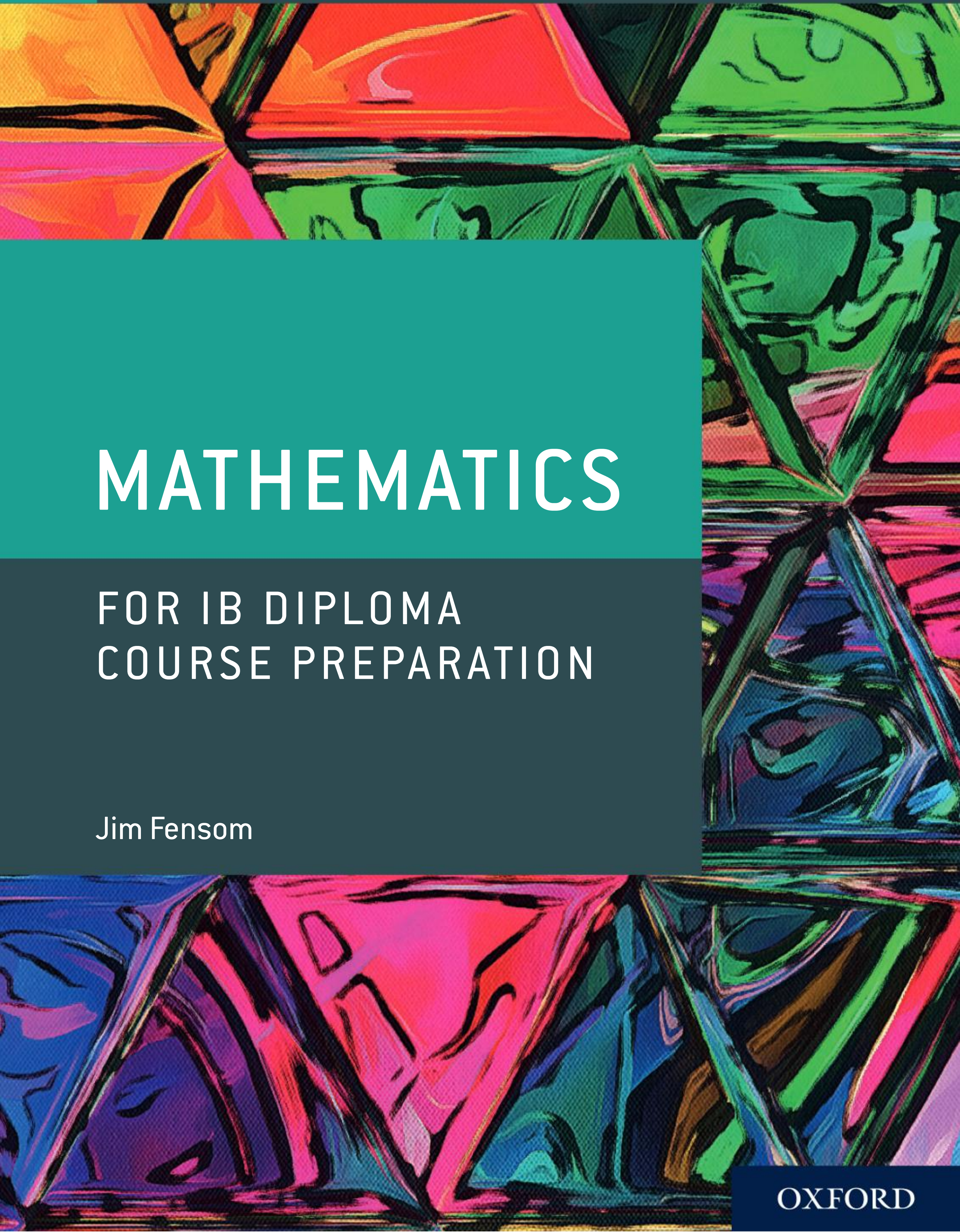

OXFORD IB COURSE PREPARATION

MATHEMATICS

FOR IB DIPLOMA COURSE PREPARATION

Jim Fensom

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Contents

Worked solutions can be found at **www.oxfordsecondary.com/9781382004923**

Introduction

The Diploma Programme (DP) is a two-year pre-university course for students in the 16–19 age group. In addition to offering a broad-based education and in-depth understanding of selected subjects, the course has a strong emphasis on developing intercultural competence, open-mindedness, communication skills and the ability to respect diverse points of view.

You may be reading this book during the first few months of the Diploma Programme or working through the book as a preparation for the course. You could be reading it to help you decide whether the Maths course is for you. Whatever your reasons, the book acts as a bridge from your earlier studies to DP Maths, to support your learning as you take on the challenge of the last stage of your school education.

DP course structure

The DP covers six academic areas, including languages and literature, humanities and social sciences, mathematics, natural sciences and creative arts. Within each area, you can choose one or two disciplines that are of particular interest to you and that you intend to study further at the university level. Typically, three subjects are studied at higher level (HL, 240 teaching hours per subject) and the other three at standard level (SL, 150 hours).

In addition to the selected subjects, all DP students must complete three core elements of the course: theory of knowledge, extended essay, and creativity, action, service.

Theory of knowledge (approximately 100 teaching hours) is focused on critical thinking and introduces you to the nature, structure and limitations of knowledge. An important goal of theory of knowledge is to establish links between different areas of shared and personal knowledge and make you more aware of how your own perspective might differ from those of others.

The **extended essay** is a structured and formally presented piece of writing of you to 4,000 words based on independent research in one of the approved DP disciplines. It is also possible to write an interdisciplinary extended essay that covers two DP subjects. One purpose of the extended essay activity is to develop the high-level research and writing skills expected at university.

Creativity, action, service involves a broad range of activities (typically 3–4 hours per week) that help you discover your own identity, adopt the ethical principles of the IB and become a responsible member of your community. These goals are achieved through participation in arts and creative thinking (creativity), physical exercises (activity) and voluntary work (service).

DP Mathematics syllabus

Basics

Two Mathematics subjects are offered in Group 5 of the IB Diploma: Mathematics: analysis and approaches (MAA) and Mathematics: applications and interpretation (MAI). For your IB Diploma course you will need to select one Mathematics course from:

- Mathematics: Analysis and Approaches (MAA) Standard Level (SL)
- Mathematics: Analysis and Approaches (MAA) Higher Level (HL)
- Mathematics: Applications and Interpretation (MAI) Standard Level (SL)
- Mathematics: Applications and Interpretation (MAI) Higher Level (HL)

The whole content of each SL course forms part of the corresponding HL course. There is also considerable overlap in the content of MAA and MAI and, indeed, 60 teaching hours is common to all four syllabuses.

One of the purposes of this book is to help guide you to the most appropriate course for you.

The courses

The MAA course follows a traditional approach to High School mathematics. The course guide describes a typical MAA student as one who "should be comfortable in the manipulation of algebraic expressions and enjoys the recognition of patterns and understands the mathematical generalization of these patterns … and (will have) the ability to understand simple proof." (IB Mathematics course guide 2019)

The MAI course puts more emphasis on the mathematics used in the workplace or in those other academic disciplines which increasingly rely on mathematics to underpin the work they do – for example, Biology, Environmental science, Economics, Medicine, Sociology. The course guide describes a typical MAI student as one who "should enjoy seeing mathematics used in real-world contexts and being used to solve real-world problems." (IB Mathematics course guide 2019)

How the book will help you to choose a course

This book covers the prior learning required for all the courses and the questions will help you develop the necessary skills and techniques for whichever course you do. There are a few sections in the book that are based on prior learning required for the HL courses. These are clearly labeled as "Higher level". You do not need to cover these sections if you are preparing for an SL course.

To help you decide which of the two subjects – MAA or MAI – might suit you best, some questions are labeled as **DP Style: Analysis and approaches** or **DP Style: Applications and interpretation**. There is a lot of overlap in the style as well as the content of the two courses, but the questions here focus on the differences.

The DP style: MAA questions will often allow you to work at a slightly more abstract level, to use algebra to generalize results, to make conjectures and devise simple proofs.

The DP style: MAI questions are mainly set in a real-world context. You will need to decide how to use the mathematics you have learned to solve the real-world problem. Then, when you have worked out the mathematics, you will need interpret your answer within the context given in the question.

Both of the higher level courses, and both of the standard level courses, are seen by the IB as equally challenging. There is, however, a considerable increase in difficulty between each standard level course and the corresponding higher level course. This is an increase in the difficulty of the concepts covered, as well as in the complexity of the mathematics and the depth of understanding required to succeed on the course.

In order to help you to decide whether or not a higher level course is for you, the DP-style questions are labeled as either SL or HL. These are questions based on the work within the prior learning section of the syllabus, but they are also designed to help you to develop these ideas and allow you to explore some of the consequences of the mathematics.

A potential higher level candidate is not expected to complete all of these DP style HL questions easily, but they should enjoy trying them. The IB guide describes a potential higher level student as someone who "enjoys spending time with problems and who gets pleasure and satisfaction from (their solution)".

Each chapter ends with a test. The test includes a DP-style question for both of the courses and for each of the levels.

Your experience of tackling the different types of question will indicate which course would be the most rewarding for you. This though is only one of several factors in deciding. You also need to consider which courses are offered by your school, which other diploma courses you are taking and any entry requirements for particular universities.

Internal Assessment

All DP subjects have an Internal Assessment (IA) component. This assessment allows students to demonstrate their knowledge and to apply skills in a topic of choice often linked to personal interest. The exploration is a piece of work that involves in-depth investigation in a chosen area of mathematics and /or its application to a real-world problem.

Although the IA has no time limitation like externally-assessed components, schools will have a timeline and internal deadlines for submission of a draft and the final submission after written feedback has been given on the draft. It is internally assessed by the teacher against five criteria, but like all DP internal assessment components it is subject to external moderation.

Using this book effectively

Throughout this book you will encounter separate text boxes to alert you to ideas and concepts. Here is an overview of these features and their icons:

Icon	Feature	Description of feature
🔑	Key terms	Mathematical terms that you need to learn to prepare you for the DP Mathematics course.
◎	Key point	Definition or rule.
🔍	Investigation	A mathematical investigation, with step-by-step instructions. This will help develop your mathematical understanding of a topic and your inquiry skills. It also prepares you for your DP Mathematics course, which is taught through investigations.
✎	Worked example	A question with a full worked solution. The working and answer are in the left-hand column. Notes in the right-hand column explain the steps in the working.
✏	Exercise	Questions for you to answer, to practise what you have learned.
➔	Note	Extra information to help you understand a worked example or an explanation.
Hint	Hint	A hint to help you answer a mathematics question.
>	Command term	Explanation of a command term – a word that tells you what you need to do in a question, for example **identify**, or **describe**.
DP style	DP style question	A question based on the mathematics in this book, written in the style of DP questions. Labelled MAI or MAA, SL or HL, to help you get an idea of the type and difficulty of questions you will be working on in each course.
Higher Level	Higher level content	Mathematics that is only required for the Higher level DP Mathematics courses.
∞	Internal link	Reference to another section in this book, where there is more information on a topic.
8	DP link	Explanation of how this topic will be used or developed further in the DP courses.
🌐	DP ready: International-mindedness	Description of the use of mathematics around the world.
🧠	DP ready: Theory of knowledge	Ideas or concepts in mathematics that prompt wider discussions about the different ways of knowing.
🎓	DP ready: Approaches to learning	Lists the skills you need to be an effective DP Mathematics learner, and that you will develop as you work through the activity.

You may have covered some of the mathematics in this book before, so you may find you do not need to spend equal amounts of time on each of the chapters.

A good place to start is the first page of each chapter where the learning outcomes and key terms to be covered are listed. You can also check the chapter summary, which comes immediately before the end-of-chapter test.

NUMBER AND ALGEBRA

1

Number

Learning outcomes

In this chapter you will learn about:

- Number systems
- Sets
- Approximation and rounding
- Absolute value
- Operations with numbers
- Prime numbers, factors and multiples
- Greatest common factor and least common multiple (HL only)
- Exponential expressions
- Exponential expressions with rational indices (HL only for MAI, SL for MAA)
- Surds (radicals)
- Rationalising the denominator (HL only)
- Standard form

Key terms

- Set
- Approximate
- Modulus
- Prime number
- Multiple
- Factor (divisor)
- Fraction
- Exponent (also called power or index)
- Surd (radical)
- Standard form

1.1 Number systems

Key point

The natural numbers, $\mathbb{N}$, are the counting numbers (including the number 0): 0, 1, 2, 3, 4, 5, …

The integers, $\mathbb{Z}$, include all the natural numbers and their negative values too: … −2, −1, 0, 1, 2, …

The rational numbers, $\mathbb{Q}$, include all the integers and numbers in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. Examples of rational numbers are $\frac{5}{6}, -\frac{1}{2}, \frac{20}{7}, 3\frac{1}{2}, \ldots$

Terminating decimals like 0.5, 175, and −7.396 and recurring decimals like 0.333… and 1.7272… are also examples of rational numbers, since they can be written in the form $\frac{a}{b}$, $b \neq 0$.

The real numbers, $\mathbb{R}$, contain both the rational and irrational numbers. Irrational numbers include π, $\sqrt{2}$, $\sqrt[3]{10}$ and any non-terminating, non-recurring decimals.

DP ready **International-mindedness**

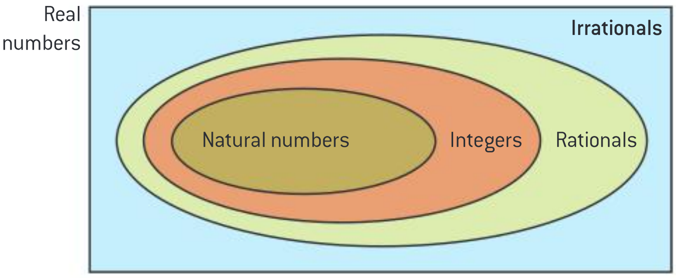
Most early number systems, such as the Babylonian, Egyptian and Roman systems, had no symbol for zero. There is evidence that the Mayan civilisation in Central America had a zero symbol in the first century BCE, the Khmer civilisation in Cambodia used a zero symbol in the 6th century CE and the Hindu number system in India used a symbol for zero in the 9th century CE.

The Western number system, based on the Hindu-Arabic system, was only introduced in the 13th century CE.

DP link

In Mathematics DP, HL students also learn about a set of numbers called the complex numbers, $\mathbb{C}$.

These groups of numbers can be organised in sequence, where each group contains all of the numbers from the previous group(s): natural numbers ($\mathbb{N}$); integers ($\mathbb{Z}$); rationals ($\mathbb{Q}$) and irrationals; real numbers ($\mathbb{R}$). For example, the rational numbers contain all of the integers and all of the natural numbers.



Command term

Identify means you should choose an answer from a number of possibilities.

Describe means you should give a detailed account.

Exercise 1.1

1 **Identify** which is the *smallest* set ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$) each number belongs to.

1 12 **2** –4 **3** 0 **4** $\sqrt{5}$

5 $\sqrt{16}$ **6** $\frac{5}{7}$ **7** $-\frac{12}{4}$

1.2 Sets

Key point

A **set** is a collection of objects: for example, a collection of numbers, letters, geometrical objects or anything else.

As you saw in 1.1, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are sets of numbers.

You **describe** a set by writing a list of everything in it, or by writing a description of what is contained in the set, written inside curly brackets (or braces) { }.

For example:

$A = \{1, 2, 3, 4\}$

$B = \{a, e, i, o, u\}$

$C = \{red, green, blue\}$

$D = \{English, Chinese, History, Physics, Mathematics, Art\}$

$E = \{$Integers between 1 and 10$\}$

$F = \{$Countries in the EU$\}$

$G = \{$Planets in the Solar System$\}$

$H = \{$Irrational numbers$\}$

You call individual items in a set **elements**.

Key point

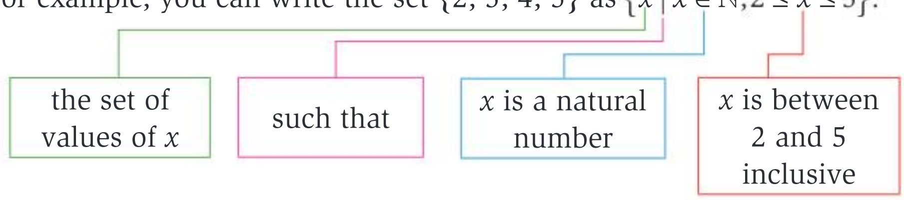
The symbol $\in$ means 'is an element of' and $\notin$ means 'is not an element of'.

For example:

$2 \in \{1, 2, 3, 4\}$ Jupiter $\in \{$Planets in the Solar System$\}$

You can also describe sets using **set-builder notation**.

For example, you can write the set $\{2, 3, 4, 5\}$ as $\{x \mid x \in \mathbb{N}, 2 \le x \le 5\}$.



Key point

The number of elements in a set A is its **cardinality**, $n(A)$.

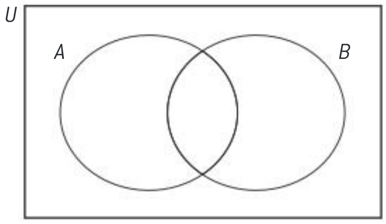
The set that contains no elements at all is the **empty set**, $\varnothing$. $n(\varnothing) = 0$.

The set of all the elements you are considering is the **universal set**, U.

Internal link

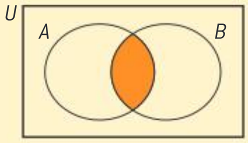
$2 \le x \le 5$ is an example of an inequality. You will study these in chapter 2.

A **Venn diagram** is a way of representing sets:

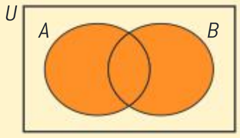


Key point

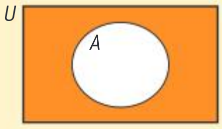
The set of elements that are in both A and B is the **intersection** of A and B, $A \cap B$.



The set of elements that are either in A or B or both is the **union** of A and B, $A \cup B$.

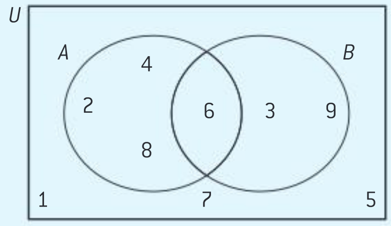


The set of elements of U that are not in A is the **complement** of A, that is, A'. $A' = \{x \mid x \in U, x \notin A\}$.



Example 1

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{3, 6, 9\}$.
Draw a Venn diagram to represent these sets.



6 is in both A and B so write it in the intersection.

The other elements of A are 2, 4 and 8. Write these in the outer part of A.

The other elements of B are 3 and 9. Write these in the outer part of B.

The other elements of U are 1, 5 and 7. Write these outside of A and B.

Exercise 1.2

1 **Write down** the set of natural numbers that are less than 6:

a as a list inside braces **b** using set-builder notation.

2 Consider the sets $P = \{2, 4, 6, 8\}$, $Q = \{1, 3, 5, 7\}$, $S = \{x \mid x \in \mathbb{Z}, 2 < x \le 6\}$.

Determine which of the following statements are true and which are false. Where a statement is false, re-write and correct it so that it is true.

a $4 \in P$ **b** $5 \notin Q$ **c** $P \cap Q = \varnothing$ **d** $n(P) = 4$ **e** $n(P) = n(S)$

DP style **Analysis and Approaches HL**

3 $U = \{n \mid n \in \mathbb{N}, n \le 10\}$, $M = \{1,2,3,5,8\}$ and $N = \{3,6,8\}$

a Draw a Venn diagram to show sets U, M and N. Write the numbers 0 to 10 in the appropriate section of the diagram.

Write down each set as a list:

b M' **c** N' **d** $M \cap N$ **e** $M \cup N$

DP link

DP students use Venn diagrams to tackle problems in probability in MAI and MAA at HL and SL.

Command term

Write down means you should obtain the answer without any calculations. You do not need to show any working.

Determine means you should obtain the only possible answer.

Note

To help you decide which of the two routes – Analysis and approaches or Applications and interpretation – might suit you best, questions have been labelled throughout. You are not expected to complete all of these DP-style questions easily, but working through them should help you to decide where your interests lie!

4 Write down the definition of rational numbers in 1.1 using set-builder notation.

DP style **Analysis and Approaches HL**

5 Given that $n(A)=6$, $n(B)=12$, $n(A'\cap B)=9$ and the universal set has 24 elements, write down:

a $n(A\cap B)$ **b** $n(A'\cap B')$ **c** $n(A'\cup B')$.

DP style **Applications and Interpretation HL**

6 A student conducts a survey of cars that pass the school. She notes the colour and the make of the cars. When looking at the data she notices the most common colour is silver and the most common make is Peugeot.

Of the 100 cars she surveyed, 38 were silver and 22 were made by Peugeot.

Given 48 were either made by Peugeot or were silver, use a Venn diagram to find:

a the number of cars that were neither made by Peugeot, nor were silver

b the number of silver Peugeots.

1.3 Approximations and rounding

When you use a calculator, the result may be more accurate than you need.

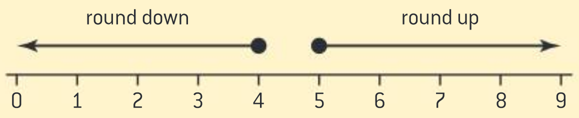
For example, Adam earns € 35 023 per year. He calculates his monthly salary:

$$35\,023 \div 12 = 2918.58333\ldots$$

He can round this to the nearest whole number of euros or he can round to 2 decimal places (d.p.) so it is in euros and cents.

Key point

When rounding, consider the figure immediately to the right of the last digit you are rounding to. If the next figure is 0, 1, 2, 3, or 4, round down. If the next figure is 5, 6, 7, 8, or 9, round up.



For Adam,

This is 5 or above

$$€\,2918\cdot583 \xrightarrow[\text{round up}]{} €\,2919 \quad \text{(nearest euro)}$$

This is below 5

$$€\,2918\cdot58\,33 \xrightarrow[\text{round down}]{} €\,2918\cdot58 \quad \text{(nearest cent)}$$

Example 2

1 Rahul earns € 35 060 per year.

- **a** **Calculate** his monthly salary.
- **b** Calculate his weekly pay.

2 Write down the value of 456.8 to the nearest whole number.

1 **a** 35 060 ÷ 12 = 2921.666… Rounding to 2 d.p. Rahul's monthly salary is € 2921.67	Round the amount in euros to the nearest cent. Look at the 3rd decimal place. Since it is 6, you round up the 2nd decimal place.
b 35 060 ÷ 52 = 674.2307… Rounding to 2 d.p. Rahul's weekly salary is € 674.23	There are 52 weeks in a year, so divide by 52. Round the amount in euros to the nearest cent. Look at the 3rd decimal place. Since it is 0, you round down the 2nd decimal place.
2 456.8 is 457 to the nearest whole number.	Look at the 1st decimal place. Since it is 8, you round the number in the units column from 6 up to 7.

Command term

Calculate means you should obtain a numerical answer showing the relevant stages in your working.

In IB examinations you should give numerical answers exactly or to three significant figures.

The first significant figure is the first non-zero digit from the left.

For example, 2.1538461538 = 2.15 to 3 s.f., 0.0215386 = 0.0215 to 3 s.f. and 40.52 = 40.5 to 3 s.f.

Example 3

Round these numbers to 3 s.f.

a 12.72 **b** 10 730 **c** 0.02646 **d** 34.65 **e** 7895

a 12.72 = 12.7 to 3 s.f.	Look at the 4th figure. Since it is 2, you round the 3rd figure down.
b 10 730 = 10 700 to 3 s.f.	The 4th figure is 3, so you round the 3rd figure down. Insert a zero to keep the place value.
c 0.02646 = 0.0265 to 3 s.f.	Begin counting from the first non-zero figure. The 4th figure is 6, so you round the 3rd figure up.
d 34.65 = 34.7 to 3 s.f.	The 4th figure is 5, so you round the 3rd figure up.
e 7895 = 7900 to 3 s.f.	The 4th figure is 5 so you would round the 3rd figure up, but since it is 9, you round up by 'adding 1' to the 3rd digit, which makes it ten.

Hint

If you are continuing a calculation and using an answer further, do not round your first answer. Only when you reach the final answer should you round to an appropriate degree of accuracy.

DP link

Consideration of errors and the effect they have on real-life situations is one of the themes of the MAI course.

Command term

Comment means you should make an observation based on the result of your calculation.

Calculator hint

Your calculator has a key marked Ans. You can press this to use the most recently calculated value to the full accuracy of your calculator. As a shortcut, for example, instead of pressing the Ans key and typing × 5, you can type × 5 and the calculator will insert Ans for you. Some calculators also let you copy and paste answers. All calculators let you store values to use later in a calculation.

Using the value to the full accuracy of your calculator makes sure that your final answer is as accurate as possible.

Exercise 1.3

1 Round each of the following numbers given in parts **a** – **i** to

- **i** 2 d.p.
- **ii** the nearest whole number
- **iii** 3 s.f.

a	764.382	**b**	234.368	**c**	0.02379
d	0.005456	**e**	15.098	**f**	86.798
g	178867.352	**h**	0.5798	**i**	29.891

DP style **Applications and Interpretation HL**

2 A box contains 13 nails and costs \$2.90.

a The cost of the box goes up by \$0.50. Calculate the new price of one nail. Write down the answer given by your calculator and then round it to 3 s.f.

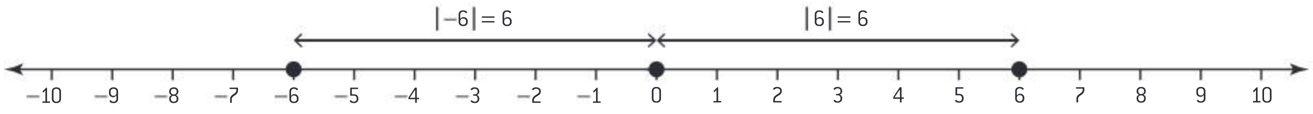
b The shopkeeper decides to put 15 nails in each box instead of 13. Use the new price of one nail from part **a** which is

- **i** given unrounded by your calculator
- **ii** rounded to 3 s.f.

to calculate the price of a box of 15 nails, correct to 2 d.p. **Comment** on what your answers to part **b** tell you about rounding before the final answer.

1.4 Modulus (or absolute value)

The modulus tells you how far away a number is from zero. It does not matter whether the number is positive or negative, so (for example) $|-6| = 6$ and $|6| = 6$.



Key point

The **modulus** or absolute value $|x|$ of x is $\begin{cases} x \text{ for } x \geq 0 \\ -x \text{ for } x < 0 \end{cases}$

Investigation 1.1

Copy and complete the table.

a	b	$\lvert a+b \rvert$	$\lvert a \rvert + \lvert b \rvert$	$\lvert a-b \rvert$	$\lvert b-a \rvert$	$\lvert a \rvert - \lvert b \rvert$	$\lvert \lvert a \rvert - \lvert b \rvert \rvert$	$\lvert a \times b \rvert$	$\lvert a \rvert \times \lvert b \rvert$
2	5								
−3	4								
1	−2								
−5	−4								

1 Look at the results for $|a + b|$ and $|a| + |b|$. What do you notice?

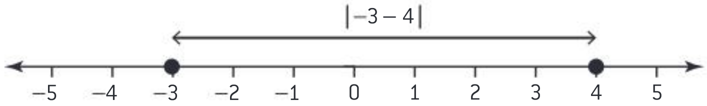
Use one of the symbols =, ≥ or ≤ (where ≤ means *less than or equal to*, and ≥ means *greater than or equal to*) to replace □ and complete the following conjecture.

$$|a+b| \square |a|+|b|$$

2 Look at the results for $|a - b|$ and $|b - a|$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$$|a-b| \square |b-a|$$



This distance is called the **absolute difference** of the two numbers.

3 Look at the results for $|a - b|$ and $\big||a|-|b|\big|$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$$|a-b| \square \big||a|-|b|\big|$$

4 Look at the results for $|a \times b|$ and $|a|\times|b|$. What do you notice?

Use one of the symbols =, ≤ or ≥ to replace □ and complete the following conjecture.

$$|a\times b| \square |a|\times|b|$$

Exercise 1.4

1 If $a = -5$, $b = 3$ and $c = -2$, **find**:

a $|ab|$ **b** $|bc|$ **c** $|abc|$ **d** $|a|\times|bc|$ **e** $|a|+|b|-|c|$ **f** $|ab|-|bc|$

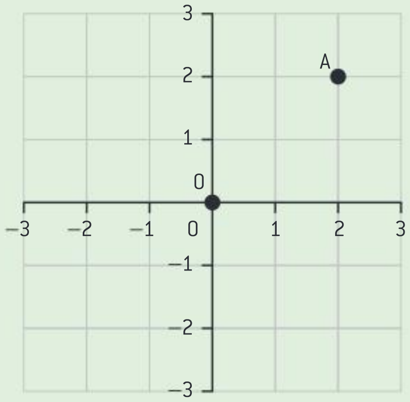
2 If $p = 4$ and $q = -10$, find:

a $\left|\frac{p}{q}\right|$ **b** $\left|\frac{q}{p}\right|$ **c** $\frac{|p|}{|q|}$ **d** $\frac{|q|}{|p|}$ **e** $|pq^2|$ **f** $\frac{p}{|q^2|}$

DP style **MAA and MAI HL**

3 The symbol $|a|$ is a measure of the shortest distance from a to zero.

Imagine a city built on a grid, with roads running north–south and east–west along every unit square. To get from one place to another, you can only travel along the roads; you cannot travel through the middle of a square. The point O is at the centre of the grid.



Command term

Find means you should obtain an answer, showing relevant stages in the working.

DP link

The MAA course will look at a range of proofs in mathematics. A result that has been proved is stronger than a conjecture as you are showing that the result is true in all cases, rather than just the specific examples from your investigation. Those at HL are more rigorous than at SL. Conjecture and proof are part of MAA HL paper 3 in particular.

Command term

Verify means

DP link

This question follows the HL paper 3 idea (for both the MAA and MAI courses) of taking an idea and extending it with further questions.

If A is a point on the grid, let |OA| be the shortest distance of travelling from O to A.

a If A is at (2,2), find |OA|.

b Conjecture a formula for |OA| if A is at the point (a,b).

c **Verify** your formula is true for the point A (–2,1).

In this system we define a **circle** with radius of 4 and centre (0,0) as the set of points which are a distance of 4 units from (O,O). Using this definition, a circle of radius 4 on the grid is the set of all points, A, for which $|OA| = 4$.

d Draw the circle of radius 4 on graph paper. Note: this will be a series of points rather than a continuous curve.

The diameter of a circle is defined as the longest distance between any two points on the circle.

e Verify that the diameter (d) of the circle is 2 times the radius.

The circumference (C) of the circle is the distance around all the points in the circle.

f Given that $C = pd$, find the value of p and verify this value is the same when considering a circle of a different radius.

DP ready **Theory of knowledge**

In investigations you look at a range of specific examples and then, from these examples, you try to deduce a general result. This result is called a **conjecture**.

How do you know that a conjecture is always true?

Mathematicians try to **prove** a conjecture. In the 17th century, Pierre de Fermat made a conjecture that $x^n + y^n = z^n$ has no solutions where x, y and z are all integers for values of $n \geq 3$. For three centuries, mathematicians tried to prove or disprove this. Finally, in 1993, Andrew Wiles presented a proof of the theorem at a conference in Cambridge.

1.5 Operations with numbers

How can you find the value of $2 + 3 \times 4$?

Either:	step (1): $2 + 3 = 5$	or	step(1): $3 \times 4 = 12$
	step (2): $5 \times 4 = 20$		step(2): $2 + 12 = 14$

Is the answer 20 or 14? It depends which order you carry out the operations.

The correct answer is 14. You get this when you carry out operations in the following way:

What happens when you add three numbers together? For example, $2 + 3 + 4$.

If you add 2 and 3 first, you get $(2 + 3) + 4 = 5 + 4 = 9$, but if you add 3 and 4 first then you get $2 + (3 + 4) = 2 + 7 = 9$, which is the same answer. So $2 + 3 + 4$ is not ambiguous. You do not need any parentheses to make it clear.

What happens when you multiply three numbers together? For example, $7 \times 2 \times 3$.

If you multiply 7 and 2 first, you get $(7 \times 2) \times 3 = 14 \times 3 = 42$, but if you multiply 2 and 3 first then you get $7 \times (2 \times 3) = 7 \times 6 = 42$, which is the same answer. So $7 \times 2 \times 3$ is not ambiguous. You do not need any parentheses to make it clear.

Multiplication and addition are examples of operations that are **associative**. The order you perform repeated operations that are associative makes no difference to the answer.

Does the same thing happen with subtraction? For example, is $(10 - 3) - 2$ the same as $10 - (3 - 2)$? Here $(10 - 3) - 2 = 7 - 2 = 5$ and $10 - (3 - 2) = 10 - 1 = 9$, so the answers are not the same.

Key point

You can summarize the rules for the order of operations as:

1. **B**rackets (or parentheses),
2. **O**rders (indices or exponents),
3. **M**ultiplication/**D**ivision,
4. **A**ddition/**S**ubtraction,

You can remember this using the mnemonic **BOMDAS**

Subtraction is not associative, so $10 - 3 - 2$ *is* ambiguous. Here you should work from left to right, but to make the calculation clear you should also include parentheses.

Division is not associative either. $(12 \div 6) \div 2 = 2 \div 2 = 1$ and $12 \div (6 \div 2) = 12 \div 3 = 4$. These answers are not equal, so you should use parentheses to make repeated division clear.

Example 4

Calculate **a** $3 + 6 + 12$ **b** $2 + 4 - 3$ **c** $8 + 2 \times 3$ **d** $13 - (5 - 2)$ **e** $5 \times 2 \times 7$ **f** $18 \div (2 \times 3)$

a $3 + 6 + 12 = 9 + 12$
$= 21$

Since + is associative you can also calculate $3 + 6 + 12 = 3 + 18 = 21$

b $2 + 4 - 3 = 6 - 3$
$= 3$

The correct order is from left to right.

c $8 + 2 \times 3 = 8 + 6$
$= 14$

You should multiply before you add.

d $13 - (5 - 2) = 13 - 3$
$= 10$

Calculate the subtraction inside the parentheses first.

e $5 \times 2 \times 7 = 10 \times 7$
$= 70$

Since × is associative the order of calculation makes no difference to the answer. It is slightly easier to multiply $5 \times 2 = 10$ first because it is easier to multiply by 10 than it would be to multiply by 14 if you had multiplied 2×7 first.

f $18 \div (2 \times 3) = 18 \div 6$
$= 3$

You can write a division using a fraction line. $18 \div (2 \times 3)$ is the same as $\frac{18}{2 \times 3}$. The fraction line takes the place of the parentheses. Take care with calculations like this.

Exercise 1.5

1 Calculate

a $3 + 4 - 2$ **b** $10 - 5 - 2$ **c** $30 \div 6 \times 2$ **d** $36 \div 4 \div 3$

2 Find

a $8 - 4 \div 2$ **b** $15 \times 2 + 4$ **c** $7 - 2 \times 3$ **d** $9 - 10 \div 5 + 2$

3 Calculate

a $6 \times (5 - 3)$ **b** $3(4 + 2) \div 2$ **c** $\frac{24}{3 \times 4}$ **d** $\frac{5 - 2 \times 3}{3 \times 2 - 7}$

When you use your calculator, entering calculations in the same way as you would write them on paper can help a lot. Your GDC will then use the correct order of operations.

Use the fraction template that looks like $\frac{\square}{\square}$ to enter a fraction line. Find out how to do this with your GDC.

Internal link

Return to **Exercise 1.5** and calculate each of the answers with a GDC. Check that you get the same results as when you calculated them by hand.

1.6 Prime numbers

When you learned multiplication tables, $1 \times 2 = 2$, $2 \times 2 = 4$, $3 \times 2 = 6, \ldots$ you were learning about multiples. The result of multiplying one positive integer by another is its **multiple**. So, the multiples of 2 are $2, 4, 6, \ldots$

The number 12 is a multiple of 1, 2, 3, 4, 6 and 12.

A **factor** (or **divisor**) is a positive integer that will divide exactly into another integer. So, 1, 2, 3, 4, 6 and 12 are all **factors** of 12.

Command term

List means you should give a sequence of brief answers with no explanation.

Key point

A **prime number** is a positive integer, greater than 1, that has exactly two factors. It is not a multiple of any other number apart from 1 and itself. Numbers that have more than two factors are composite. The number 1 is neither prime nor composite.

Example 5

a **List** the first 8 multiples of 3.

b Find all the factors of 48.

a The multiples of 3 are:

3, 6, 9, 12, 15, 18, 21, 24.

b $48 = 1 \times 48, 2 \times 24, 3 \times 16, 4 \times 12, 6 \times 8$

The factors of 48 are:

1, 2, 3, 4, 6, 8, 12, 16, 24, 48

Example 6

Which of these numbers is prime?

a 17 **b** 9 **c** 51 **d** –11 **e** 0

17 is prime	$17 = 17 \times 1$ are the only two factors
9 is not prime	$9 = 9 \times 1$ $9 = 3 \times 3$. It has more than two factors
51 is not prime	$51 = 51 \times 1$ $51 = 17 \times 3$
–11 is not prime	–11 is not positive
0 is not prime	0 is not a positive integer

Investigation 1.2

In this investigation, you will find all the prime numbers between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1 Draw a circle around 2. Cross through all multiples of 2 (such as 4, 6, 8, 10,…).

2 3 is the next number that you have not marked. Draw a circle around 3 and cross through all of its multiples. You have already crossed some of these.

3 Continue through the table, circling the first unmarked number and crossing through all its multiples.

4 What is the largest circled number that has a multiple in the square?

5 Circle the remaining unmarked numbers that are greater than 1.

6 List all the circled numbers: 2, 3, … etc. These are the prime numbers.

The number 1 is still unmarked. It is neither a prime nor a composite number.

7 Find the values of x^2+x+11 when $x = 0, 1, 2, \ldots$ to complete this table

x	0	1	2	3	4	5	6	7	8	9	10
x^2+x+11	11	13									

8 The values when $x = 0$ and when $x = 1$ are both prime numbers. Which other values of x give prime values? Which is the smallest composite number?

9 Does the formula x^2+x+11 find all the prime numbers up to the first composite number?

Exercise 1.6

1 Are these numbers prime or composite?

a 113 **b** 251 **c** 119 **d** 173 **e** 169

DP style **Analysis and Approaches**

2 The operation $\cdot$ is defined for $a,b \in \mathbb{Z}$ by $a\cdot b = a+b-2$

a By considering $a\cdot b\cdot c$ show that $\cdot$ is associative.

The identity element e is such that $a\cdot e = a$.

b Find the value of e.

c Find **i** $a\cdot a$ **ii** $a\cdot a\cdot a$ **iii** $a\cdot a\cdot a\cdot a$

d Suggest an expression for $\underbrace{a\cdot a\cdot a\cdot\ldots\cdot a\cdot a\cdot a}_{n\text{ terms}}$

e Hence, find $a\cdot 2\cdot a\cdot 2\cdot a\cdot 2\cdot a\cdot 2\cdot a\cdot 2\cdot a$

DP link

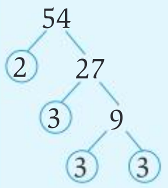
This investigation is similar to the style of investigation which you will encounter if you study the MAA course. You will look at a variety of specific examples and then try to use these to suggest a generalization or rule.

Higher Level

1.7 Greatest common factor and least common multiple

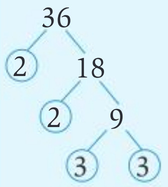
Example 7

Find the greatest common factor of 54 and 36.



$54 = 2\times3\times3\times3$

Begin dividing each number by the smallest prime number which is a factor. Here, divide by each of the prime numbers 2, 3, … in turn until you reach an answer 1.



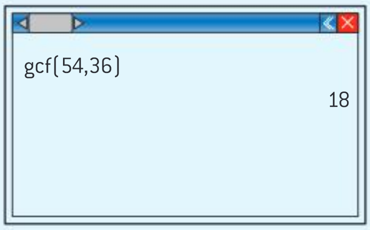
$36 = 2\times2\times3\times3$

Write each number as a **product** of the divisors.

$2\times3\times3 = 18$

The greatest common factor of 54 and 36 is 18.

Find the product of all the factors that are common to both numbers.



gcf(54,36)

18

Most GDCs have a function that will find the greatest common factor.

Key point

The **greatest common factor** (divisor) of two (or more) numbers is the largest number that will divide into them both.

Note

A **product** is two or more numbers multiplied together.

Key point

The **least common multiple** of two (or more) numbers is the smallest number that both (or all) the numbers will divide into.

DP link

In Maths DP, HL students also learn to generalize results of greatest common factor and least common multiple in algebra and use them for factorization and combining algebraic fractions.

Example 8

Find the least common multiple of 15 and 25.

The multiples of 15 are:

15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240, …

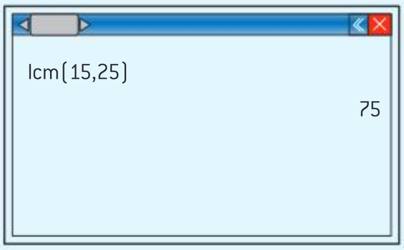
The multiples of 25 are:

25, 50, 75, 100, 125, 150, 175, 200, 225, 250, …

List multiples of each number

The least common multiple of 15 and 25 is 75.

Find the smallest number that is in each list.



```
lcm(15,25)
                    75
```

Most GDCs have a function that will find the least common multiple.

Exercise 1.7

1 Find the greatest common factor of

a 36 and 20 **b** 6 and 12 **c** 18 and 42 **d** 36, 54 and 90.

2 Find the least common multiple of

a 6 and 12 **b** 8 and 20 **c** 15 and 40 **d** 4, 6 and 21.

DP style **Applications and Interpretation HL**

3 The size of populations of different species often follows a periodic cycle. Often the population will increase to a certain level and then decrease due to competition. The length of the cycle will be the time between successive maximums.

Suppose three species have cycles of length 4, 6 and 9 years respectively, and suppose all populations are at their maximum at year 0.

a Find when the populations will again all be at their maximum together.

Periodical cicadas emerge in a swarm after a fixed number of years. Assume the cicadas are eaten by all three of the species above.

b If a population of cicadas emerged in year 0 and periodically every 12 years after that, how many of the next ten emergences would match with **i** one **ii** two **iii** three maximums of the predator populations.

In fact, the number of years between emergences for populations of periodic cicadas are almost always prime numbers.

c Repeat part **b** given that the population emerges every 13 years.

d Two species of periodic cicadas share the same territory. The periods of the two groups are 13 and 17 years. Explain why these numbers will provide an evolutionary advantage over the similar length cycles of 12 and 18 years.

DP style **Analysis and Approaches HL**

4 **a** **i** Write 30 and 135 as products of their prime factors.

ii By consideration of the factors, show that the product of 30 and 135 is equal to the product of their highest common factor and least common multiple.

Consider two numbers $m, n \in \mathbb{Z}$. Let h be the highest common factor of m and n and l be the least common multiple of m and n.

b Show that $m \times n = h \times l$

1.8 Fractions

Although you can perform these numerical calculations with a calculator, your GDC cannot perform operations with algebraic fractions in the same way. It is therefore important that you understand how to perform these operations without your GDC so you know the methods required if you start calculating with algebraic fractions.

The way you write a fraction is not unique. For every fraction there are **equivalent fractions**. Equivalent fractions have the same value but have different denominators, such as $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$. The fraction you should use is one that is in its **simplest terms**, that is, a fraction that you have cancelled as far as you can.

DP link

In Maths DP, HL students also learn to manipulate algebraic fractions.

Example 9

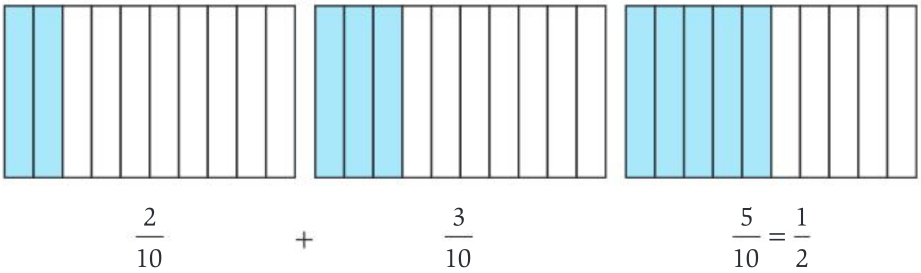
a Find fractions that are equivalent to $\frac{2}{3}$.

b Write $\frac{60}{108}$ in its simplest terms.

a $\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$	Multiply the numerator and denominator by the same number to find an equivalent fraction.
b $\frac{60}{108} = \frac{60 \div 2}{108 \div 2} = \frac{30}{54} = \frac{30 \div 2}{54 \div 2} = \frac{15}{27} = \frac{15 \div 3}{27 \div 3} = \frac{5}{9}$ write this as $\frac{\cancel{60}}{\cancel{108}} = \frac{\cancel{30}}{\cancel{54}} = \frac{\cancel{15}}{\cancel{27}} = \frac{5}{9}$	Divide by common factors until no more dividing is possible. This is called 'cancelling'.

When you add (or subtract) fractions, they must have the same denominator. You should write any answer in its simplest terms.

For example: $\frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$

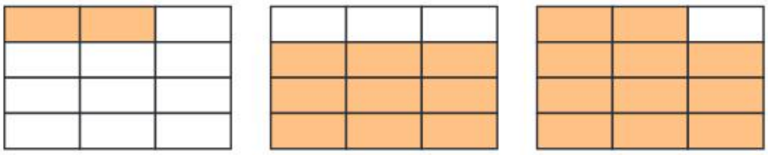


DP ready **International-mindedness**

The Egyptians wrote fractions as a sum of fractions which all had a numerator of 1. For example, you can write $\frac{2}{3}$ as $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$. The Babylonians, who lived in present-day Iraq, used a sexagesimal system based on 60. They had a method similar to our decimal system. $0.11=\frac{1}{10}+\frac{1}{100}$ is equivalent to $\frac{6}{60}+\frac{36}{60^2}$. The present-day system of writing fractions was not introduced in Europe until the 17th century.

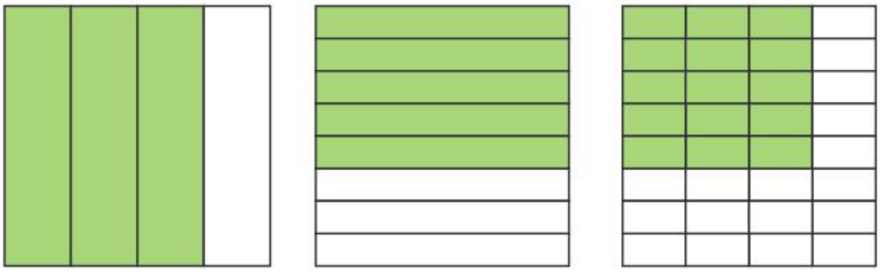
If you are adding fractions that do not have the same denominator, you must first find equivalent fractions with a **common denominator**.

For example, $\frac{1}{6}+\frac{3}{4}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$.



$$\frac{1}{6}\left(\frac{2}{12}\right) + \frac{3}{4}\left(\frac{9}{12}\right) = \frac{11}{12}$$

Multiplication of fractions is the process of finding one fraction of another. For example, to find $\frac{3}{4}\times\frac{5}{8}$ you have to find $\frac{3}{4}$ of $\frac{5}{8}$.



$$\frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$$

You can see in the diagram above that you multiply fractions by multiplying the numerators together and multiplying the denominators together.

For example, $\frac{3}{4}\times\frac{5}{8}=\frac{3\times5}{4\times8}=\frac{15}{32}$

Because numerators are multiplied, and denominators are multiplied, you can cancel fractions 'diagonally' before multiplication in order to make the calculation simpler.

Find $\frac{1}{6}\times\frac{4}{5}$.

Dividing top and bottom by 2 gives $\frac{1}{\cancel{6}_3}\times\frac{\cancel{4}^2}{5}=\frac{1\times2}{3\times5}=\frac{2}{15}$.

Dividing by 2 is the same as multiplying by $\frac{1}{2}$ and dividing by 4 is the same as multiplying by $\frac{1}{4}$. It follows that dividing by $\frac{1}{2}$ is the same as multiplying by 2 and dividing by $\frac{1}{4}$ is the same as multiplying by 4. $\frac{1}{2}$ is called the **reciprocal** of 2 and $\frac{1}{4}$ is the reciprocal of 4.

What happens if you divide by $\frac{3}{4}$?

First, split this into dividing by 3 and dividing by $\frac{1}{4}$. This is the same as multiplying by $\frac{1}{3}$ and multiplying by 4. Putting these back together, this is the same as multiplying by $\frac{4}{3}$; the reciprocal of $\frac{3}{4}$.

For example, $\frac{7}{8}\div\frac{3}{4}=\frac{7}{\cancel{8}_2}\times\frac{\cancel{4}^1}{3}=\frac{7\times1}{2\times3}=\frac{7}{6}$.

Key point

Division is equivalent to multiplication by a reciprocal.

The answer to this example is what is known as an **improper fraction** as the numerator is greater than the denominator. You can write the answer as a **mixed number**, $\frac{7}{6} = 1\frac{1}{6}$.

Exercise 1.8

1 Write in simplest terms

a $\frac{12}{28}$ **b** $\frac{15}{40}$ **c** $\frac{18}{54}$ **d** $\frac{125}{1000}$

2 Calculate

a $\frac{3}{11} + \frac{5}{11}$ **b** $\frac{1}{3} + \frac{1}{6}$ **c** $\frac{3}{5} - \frac{1}{4}$ **d** $\frac{3}{8} + \frac{5}{12}$

3 Calculate

a $\frac{1}{2} \times \frac{3}{5}$ **b** $\frac{2}{3} \times \frac{9}{16}$ **c** $\frac{3}{4} \div \frac{5}{8}$ **d** $\frac{1}{2} \div \frac{1}{8}$

1.9 Exponential expressions

An **exponent** (or **index**) is a **power** of a number.

3 is the exponent

$$8^3 = \underbrace{8 \times 8 \times 8}_{3\text{ times}}.$$

The exponent is the number of times you multiply the number by itself.

You can write:

$2 \times 2 \times 2 \times 2$ as 2^4

3×3 as 3^2

$10 \times 10 \times 10 \times 10 \times 10 \times 10$ as 10^6.

You can also write 2 as 2^1 or 4 as 4^1.

Key point

$a^1 = a$

$a^n = \underbrace{a \times a \times \cdots \times a}_{n\text{ times}}$

Investigation 1.3

1 $2^1 \times 2^2 = (2) \times (2 \times 2) = 2 \times 2 \times 2 = 2^3$

Multiply $2^2 \times 2^3$, $2^1 \times 2^4$, $2^2 \times 2^2$, $2^3 \times 2^4$ writing your answers as powers of 2.

Do you notice a pattern? Can you generalize and find a rule for combining the powers of 2 when you multiply?

Multiply $3^2 \times 3^3$ and $5^1 \times 5^2$.

Does your rule apply to these multiplications as well?

2 $2^5 \div 2^2 = \frac{2 \times 2 \times 2 \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2}} = 2 \times 2 \times 2 = 2^3$

Divide $2^4 \div 2^3$, $2^6 \div 2^4$, $2^3 \div 2^2$, $2^5 \div 2^1$ writing your answers as powers of 2.

Is there a pattern? Can you find a rule for division with powers of 2?

Try dividing some powers of 3 and 5. Does this rule apply to division with powers of these numbers too?

Key point

You can combine indices according to these rules:

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$\left(a^m\right)^n = a^{m\times n}$$
$$a^n \times b^n = \left(a \times b\right)^n$$

DP link

In DP Maths, HL and SL students learn about logarithms, which rely on these laws of indices.

3 $\left(2^2\right)^3 = (2\times 2)\times(2\times 2)\times(2\times 2) = 2\times 2\times 2\times 2\times 2\times 2 = 2^6$

Calculate these powers of powers: $\left(2^1\right)^4$, $\left(2^2\right)^2$, $\left(2^3\right)^2$, writing your answers as powers of 2.

Is there a pattern? Can you generalize and find a rule for powers of powers of 2?

Try this with powers of other numbers. Does the rule you found apply to these numbers too?

4 $$\begin{aligned} 2^3 \times 3^3 &= (2\times 2\times 2)\times(3\times 3\times 3)\\ &= 2\times 2\times 2\times 3\times 3\times 3\\ &= 2\times 3\times 2\times 3\times 2\times 3\\ &= (2\times 3)\times(2\times 3)\times(2\times 3)\\ &= 6\times 6\times 6\\ &= 6^3 \end{aligned}$$

Calculate $2^2 \times 4^2$, $3^2 \times 4^2$, $2^4 \times 5^4$

Is there a pattern? Can you generalize and find a rule for multiplying the same power of two numbers?

Example 10

Find the value of each expression. Where possible, use the laws of indices to first simplify the expression.

a $2^3 \times 2^4$ **b** $3^2 \times 4^2$ **c** $2^2 \times 3^2 \times 2^3 \times 3^3 \times 5^2$ **d** $6^5 \div 6^3$

e $\dfrac{2^4 \times 3^2 \times 3^3}{2^2 \times 3^4}$ **f** $(3^2)^2$ **g** $\sqrt{225}$ **h** $\sqrt[3]{27}$

a $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$	Add indices when multiplying.
b $3^2 \times 4^2 = (3\times 4)^2 = 12^2 = 144$	Same index.
c $\begin{aligned} 2^2\times 3^2\times 2^3\times 3^3\times 5^2 &= 2^{2+3}\times 3^{2+3}\times 5^2\\ &= 2^5\times 3^5\times 5^2\\ &= (2\times 3)^5\times 25\\ &= 6^5\times 25\\ &= 7776\times 25\\ &= 194\ 400 \end{aligned}$	Deal with powers of different numbers separately.
d $6^5 \div 6^3 = 6^{5-3} = 6^2 = 36$	Subtract indices when dividing.
e $\dfrac{2^4\times 3^2\times 3^3}{2^2\times 3^4} = 2^{4-2}\times 3^{2+3-4} = 2^2\times 3^1 = 4\times 3 = 12$	A fraction line implies division.
f $(3^2)^2 = 3^{2\times 2} = 3^4 = 81$	Multiply the indices.
g $\sqrt{225} = 15$	$\sqrt{225}$ is the number which when squared gives 225. The value can be found using the GDC.
h $\sqrt[3]{27} = 3$	$\sqrt[3]{27}$ is the number which when cubed gives 27. Again, the value can be found using the GDC.

Exercise 1.9a

1 Find:

a 2^5 b 3^3 c 10^6

d $(-3)^5$ e -4^2

2 Calculate:

a $5^2 \times 5^2$ b $4^6 \div 4^4$ c $2^3 \times 5^2 \times 2^2 \times 5^3$

d $\left(2^3\right)^4$ e $\dfrac{3^2 \times 3^4}{3^3}$ f $\dfrac{2^2 \times 3^3}{2 \times 3}$

g $\dfrac{4^3 \times 5^4}{4^4 \times 2 \times 5^2}$ h $\left(7^2\right)^3 \div 7^4$

3 Use a calculator to find:

a 1.4^6 b 2.53^5 c 1.025^{14}

d $(-0.3)^5$ e $(-2.5)^4$

DP link

DP students will study the laws of exponents in greater detail in both MAA and MAI at SL and HL

Higher Level

So far in section **1.9**, you have only dealt with indices that are positive integers.

Look at 2^0. Applying the rules of indices, $2^2 \div 2^2 = 2^{2-2} = 2^0$. **Compare** this to $4 \div 4 = 1$. It follows that $2^0 = 1$.

What does 2^{-1} mean? Applying the rules of indices,

$$\begin{aligned} 2^1 \div 2^2 &= 2^{1-2}. \\ &= 2^{-1} \end{aligned}$$

Compare this to $2 \div 4 = \frac{1}{2}$.

You can **deduce** that $2^{-1} = \frac{1}{2}$.

Now look at $2^{\frac{1}{2}}$. What does this mean? Applying the rules of indices,

$$\begin{aligned} 2^{\frac{1}{2}} \times 2^{\frac{1}{2}} &= 2^{\frac{1}{2}+\frac{1}{2}} \\ &= 2^1 \\ &= 2 \end{aligned}$$

Compare this to $\sqrt{2} \times \sqrt{2} = 2$.

You can deduce that $2^{\frac{1}{2}} = \sqrt{2}$.

Similarly $2^{\frac{1}{3}} = \sqrt[3]{2}$

Generalizing from these results:

Command term

Compare means you should give an account of the similarities between two items, referring to both of them throughout.

Deduce means you should reach a conclusion from the information given.

Internal link

You use this result to express numbers in standard form.

Example 11

Find:

a 10^{-3} b $4^{-2} \times 4^3$ c $3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ d $\dfrac{2^3 \times 2^{-4}}{2^{-1}}$

a $10^{-3} = \dfrac{1}{10^3} = \dfrac{1}{1000} = 0.001$

b $4^{-2} \times 4^3 = 4^{-2+3} = 4^1 = 4$

Key point

$a^0 = 1$

$a^{-n} = \dfrac{1}{a^n}$

$a^{\frac{1}{n}} = \sqrt[n]{a}$

c $3^{\frac{1}{2}} \times 2^{\frac{1}{2}} = (3 \times 2)^{\frac{1}{2}} = \sqrt{6}$	Both numbers are to the same power.
d $\dfrac{2^3 \times 2^{-4}}{2^{-1}} = 2^{3-4-(-1)} = 2^0 = 1$	Subtract the negative index to divide.

Key point

$$a^{\frac{m}{n}} = {a^{\frac{1}{n}}}^{m} = \left(\sqrt[n]{a}\right)^m$$

$$= \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

Look at $2^{\frac{3}{2}}$. Since you can write $\frac{3}{2} = \frac{1}{2} \times 3$

$2^{\frac{3}{2}} = {2^{\frac{1}{2}}}^{3}$ or $2^{\frac{3}{2}} = \left(2^3\right)^{\frac{1}{2}}$

$= \left(\sqrt{2}\right)^3$ $= \sqrt{2^3}$

Generalizing from this result we get the key point here.

Example 12

Find $8^{\frac{4}{3}}$

$8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16$	Notice that it is easier to calculate $\left(\sqrt[3]{8}\right)^4$ than $\sqrt[3]{8^4} = \sqrt[3]{4096}$

Exercise 1.9b

1 Calculate:

a 3^{-2} **b** 5^{-1} **c** 4^{-3} **d** $\left(\frac{1}{2}\right)^{-1}$ **e** $\left(\frac{2}{3}\right)^{-2}$

2 Find:

a $5^{-2} \times 2^4$ **b** $3^{-1} \times 3$ **c** $12 \times 3^{-1} \times 2^{-2}$ **d** $\dfrac{5^2}{5^4}$ **e** $\dfrac{2^3 \times 3^2}{2^5 \times 3^3}$

3 Calculate:

a $27^{\frac{1}{3}}$ **b** $\left(\frac{16}{9}\right)^{\frac{1}{2}}$ **c** $4^{\frac{5}{2}}$ **d** $32^{\frac{3}{5}}$ **e** $9^{-\frac{1}{2}}$

DP style **Analysis and Approaches HL**

4 **a** **i** Find the greatest common factor of 90 and 135.

ii Hence prove $3^{135} > 5^{90}$

b Find a counter example to show that the statement $\left(a^n\right)^{\frac{1}{n}} = a$ is not always true.

DP style **Applications and Interpretation HL**

5 Kepler's third law of planetary motion states that the average distance of a planet from the sun (R), where R is measured in astronomical units (AU), is related to the period (T) of its orbit by the equation

$$R^3 = kT^2$$

where T is time in days, and k is a constant.

a Given that the average distance of the earth from the sun is one astronomical unit (AU) and that the earth takes 365.25 days to orbit the sun, find the value of k.

b The orbital period of Jupiter is 4333 days. Find its average distance from the sun in astronomical units.

c Rewrite the equation in the form $T = k_1 R^a$

d Hence find the orbital period of Neptune given its average distance from the sun is 30.07 AU.

1.10 Surds (radicals)

Numbers like $\sqrt{2}, \sqrt{3}$ and $\sqrt{7}$ are **surds** (or **radicals**). Surds are irrational numbers.

Key point

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab} \text{ and } \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Note

Square numbers, such as $\sqrt{4}$, are not surds as, for example, $\sqrt{4} = 2$.

Investigation 1.4

Look at these expressions:

$$\sqrt{2} \times \sqrt{6} \qquad \frac{6}{\sqrt{3}} \qquad \frac{12}{\sqrt{12}} \qquad \frac{12}{2\sqrt{3}} \qquad \frac{12-6\sqrt{3}}{2-\sqrt{3}} \qquad \sqrt{12}$$

$$\frac{6+2\sqrt{3}}{1+\sqrt{3}} \qquad 2\sqrt{3} \qquad \frac{6\sqrt{2}}{\sqrt{6}} \qquad \frac{2\sqrt{6}}{\sqrt{2}} \qquad \frac{12}{\sqrt{2} \times \sqrt{6}}$$

1 Can you find any that are equal? If necessary, use your calculator to evaluate them. (Some GDCs will evaluate the expressions as decimals while others will express them in terms of surds unless you use the keypress that gives a decimal answer).

2 Which of the expressions that are equal do you think are the simplest?

3 Calculate $2\sqrt{3}$ to 3 s.f. and square your answer. Calculate $\left(2\sqrt{3}\right)^2$. Compare your answers. Which is the most accurate?

You should write expressions containing surds so that you cannot simplify them any further.

DP link

This style of investigation is similar to those contained in both the MAI and MAA courses.

Example 14

Simplify:

a $\sqrt{18}$ **b** $\sqrt{6} \times \sqrt{8}$

a $\sqrt{18} = \sqrt{2 \times 9} = \sqrt{2} \times \sqrt{9} = \sqrt{2} \times 3 = 3\sqrt{2}$	Look for factors that are square numbers
b $\sqrt{6} \times \sqrt{8} = \sqrt{6 \times 8} = \sqrt{48} =$ $= \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$	Combine terms and then look for square factors

Exercise 1.10a

1 Simplify:

a $\sqrt{32}$ **b** $\sqrt{12} \times \sqrt{10}$

2 Simplify:

a $\sqrt{2}\left(3+\sqrt{2}\right)$ **b** $3\sqrt{2} + \sqrt{8} - \sqrt{18}$

DP link

DP students taking MAA at SL and HL will study exact values of trigonometric ratios using surds.

Higher Level

Rationalizing the denominator

An expression is simpler if it has a rational denominator.

To rationalize the denominator in an expression like $\frac{1}{\sqrt{2}}$ you multiply the numerator and denominator by $\sqrt{2}$.

You can also simplify more complicated expressions like $\frac{1}{1+\sqrt{2}}$ by multiplying top and bottom by the equivalent expression with opposite sign in the middle, e.g. $1-\sqrt{2}$

Example 14

Simplify **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{6}{5\sqrt{3}}$ **c** $\frac{1}{1+\sqrt{2}}$

a $\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$

Multiply numerator and denominator by $\sqrt{2}$

b $\frac{6}{5\sqrt{3}} = \frac{6 \times \sqrt{3}}{5\sqrt{3} \times \sqrt{3}} = \frac{6\sqrt{3}}{15} = \frac{2\sqrt{3}}{5}$

Multiply numerator and denominator by $\sqrt{3}$

c $\frac{1}{1+\sqrt{2}} = \frac{1}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}}$

Multiply numerator and denominator by $1-\sqrt{2}$. If $1+\sqrt{2}$ is $a+b$ then $1-\sqrt{2}$ is $a-b$.

$= \frac{1-\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2}$

Use the difference of 2 squares result (see Internal link box below)

$= \frac{1-\sqrt{2}}{1-2}$

simplify

$= \frac{1-\sqrt{2}}{-1}$

divide by -1

$= \sqrt{2}-1$

Note

The rationalized form is simpler because its relative size can easily be seen and it can then be used to add surds.

Internal link

Multiplying top and bottom by the equivalent expression with opposite sign in the middle comes from a result called the difference of two squares: $(a+b)(a-b) = a^2 - b^2$. You will learn about this in chapter 2.

Key point

To rationalize the denominator when it is $\sqrt{a} \pm \sqrt{b}$ you multiply the numerator and denominator of the fraction by $\sqrt{a} \mp \sqrt{b}$.

Exercise 1.10b

1 Simplify

a $\frac{2}{\sqrt{6}}$ **b** $\frac{\sqrt{24}}{2\sqrt{3}}$ **c** $\frac{\sqrt{3} \times \sqrt{10}}{\sqrt{12} \times \sqrt{5}}$

2 Simplify

a $\frac{1}{2+\sqrt{3}}$ **b** $\frac{9}{\sqrt{5}-\sqrt{2}}$ **c** $\frac{6}{3\sqrt{2}+2}$

d $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ **e** $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{8}+2\sqrt{3}}$

1.11 Standard form

In science, you often have to deal with very large and very small numbers.

To help make large and small numbers more comprehensible, you can use **standard form** (or **scientific notation**).

Key point

In standard form you write numbers in the form $a \times 10^n$ where $1 \le a < 10$ and $n \in \mathbb{Z}$.

Did you know

For example, the average distance from the Earth to the Sun is about 149 598 000 000 m, one atomic weight unit is 0.000 000 000 000 000 000 000 000 001 66 kg and the surface area of the earth is 453 000 000 000 000 m^2.

To be able to use standard form, you need to use powers of 10. Remember that $10^1 = 10$, $10^3 = 1000$, etc. You will also need to know that $10^0 = 1$ and that $10^{-n} = \frac{1}{10^n}$, for example $10^{-3} = \frac{1}{10^3} = 0.001$.

Numbers written in standard form are easier to compare and easier to calculate with.

Internal link

Section 1.9 explained indices or powers.

Example 15

1 Write these numbers in standard form.

a 149 598 000 000 **b** 0.000 000 000 000 000 000 000 000 001 66

c 453 000 000 000 000

a $149\,598\,000\,000 = 1.496 \times 100\,000\,000\,000$

$= 1.496 \times 10^{11}$

b $0.000\,000\,000\,000\,000\,000\,000\,000\,001\,66$

$= 1.66 \times 0.000\,000\,000\,000\,000\,000\,000\,000\,001$

$= 1.66 \times 10^{-27}$

c $453\,000\,000\,000\,000$

$= 4.53 \times 100\,000\,000\,000\,000$

$= 4.53 \times 10^{14}$

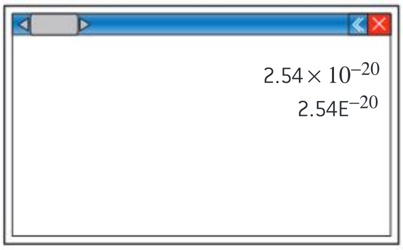
DP link

In DP HL students will learn more about indices that are not positive integers.

DP link

DP students will study operations with numbers in standard form in MAA and MAI at SL and HL.

Your calculator will express answers that are very large or very small in standard form. Some calculators use a recognizable index notation, but others use the symbol E. This form of calculator notation (seen in some computer applications too) is **not** acceptable in the DP course. For example, you should write 3.4×10^3 and **not** 3.4E3.



2.54×10^{-20}

2.54E−20

You cannot always give an answer in standard form, so you will need to know how to write it as a decimal number.

Example 16

Write these numbers as decimal numbers.

a 2.12×10^2 **b** 3.58×10^4 **c** 8.05×10^{-1} **d** 6.95×10^{-5}

a $2.12 \times 10^2 = 212$	Move the point two places to the right.
b $3.58 \times 10^4 = 35\,800$	Move the point four places to the right.
c $8.05 \times 10^{-1} = 0.805$	Move the decimal point one place to the left.
d $6.95 \times 10^{-5} = 0.0000695$	Move the point 5 places to the left and fill extra places with zeros.

The way you enter numbers in standard form also depends on the GDC you have. To enter the number 2.54×10^{-20} you should type either $2.54 \times \boxed{10^x}\ -20$ or $2.54\ \boxed{\text{E}}\ -20$, depending on which GDC you are using.

You could use the 10^x key and type $2.54 \times 10^x\ -20$ or you could use the ^ key and type 2.54 × 10 ^ −20, but the standard form key E requires only one keypress and is easier to use.

Exercise 1.11

1 Jupiter's diameter is 1.43×10^5 km and its mean distance from the Sun is 7.78×10^8 km.
Saturn's diameter is 1.21×10^5 km and its mean distance from the Sun is 1.43×10^9 km.
State which of the two planets is farthest from the Sun and which is the largest.

2 Write these numbers in standard form.

a 324 000 000 **b** 456 000 **c** 0.000 128 **d** 0.000 006 21

3 Write these numbers as decimal numbers.

a 2.50×10^3 **b** 4.81×10^1 **c** 2.85×10^{-2} **d** 3.07×10^{-4}

DP style **Applications and Interpretation SL**

4 Protons and neutrons have a mass of 1.67×10^{-27} kg and the mass of an electron is 9.11×10^{-31} kg.

a Calculate how many times more massive a proton or neutron is than an electron.

b An oxygen atom has 8 protons, 8 neutrons and 8 electrons. Calculate how much it weighs.

Chapter summary

☐ Numbers:

- ☐ The natural numbers, $\mathbb{N}$, are the counting numbers (including the number 0): 0, 1, 2, 3, 4, 5, …
- ☐ The integers, $\mathbb{Z}$, include all the natural numbers and their negative values too: … −2, −1, 0, 1, 2, …
- ☐ The rational numbers, $\mathbb{Q}$, include all the integers and numbers in the form $\frac{a}{b}$, $b \neq 0$, where a and b are integers. Examples of rational numbers are $\frac{5}{6}, -\frac{1}{2}, \frac{20}{7}, 3\frac{1}{2}, \ldots$, terminating decimals like 0.5, 1.75, −7.396, and recurring decimals like 0.333… and 1.7272… .
- ☐ The real numbers, $\mathbb{R}$, contain both the rational and irrational numbers. Irrational numbers include $\pi, \sqrt{2}, \sqrt[3]{10}$ and any non-terminating, non-recurring decimals.

☐ Sets:

- ☐ A **set** is a collection of objects, for example a collection of numbers, letters, geometrical objects or anything else.
- ☐ The number of elements in a set A is its **cardinality**, $n(A)$.
- ☐ The set that contains no elements at all is the **empty set**, $\varnothing$. $n(\varnothing) = 0$.
- ☐ The set of all the elements you are considering is the **universal set**, U.
- ☐ The set of elements of U that are not in A is its **complement**, A'. $A' = \{x \mid x \in U, x \in A\}$.
- ☐ The set of elements that are either in A or B or both is the **union** of A and B, $A \cup B$.
- ☐ The set of elements that are in both A and B is the **intersection** of A and B, $A \cap B$.

☐ When rounding, consider the figure immediately to the right of the last digit you are rounding to. If the next figure is 0, 1, 2, 3, or 4, round down. If the next figure is 5, 6, 7, 8, or 9, round up.

☐ The **modulus** or absolute value $|x|$ of x is

$$\begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

☐ You can summarize the rules for the order of operations as:

1. **B**rackets (or parentheses),
2. **O**rders (indices or exponents),
3. **M**ultiplication/**D**ivision,
4. **A**ddition/**S**ubtraction,

You can remember this using the mnemonic **BOMDAS**

☐ A **prime number** is a positive integer, greater than 1 that has exactly two factors. It is not a multiple of any other number apart from 1 and itself. Numbers that have more than two factors are composite. The number 1 is neither prime nor composite.

☐ The **greatest common factor** (divisor) of two (or more) numbers is the largest number that will divide into them both.

☐ The **least common multiple** of two (or more) numbers is the smallest number that the numbers will divide into.

☐ Fractions:

  ☐ Equivalent fractions have the same value

  ☐ If you are adding fractions that do not have the same denominator, you must first find equivalent fractions in order to write them with a **common denominator**

  ☐ When multiplying fractions, multiply the numerators and multiply the denominators

  ☐ Division is equivalent to multiplication by a reciprocal

☐ Rules of indices:

  ☐ $a^1 = a$

  $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$

  ☐ $a^m \times a^n = a^{m+n}$

  $a^m \div a^n = a^{m-n}$

  $(a^m)^n = a^{m \times n}$

  $a^n \times b^n = (a \times b)^n$

☐ $a^0 = 1$ (HL)

☐ $a^{-n} = \dfrac{1}{a^n}$ (HL)

☐ $a^{\frac{1}{n}} = \sqrt[n]{a}$ (HL)

☐ $a^{\frac{m}{n}} = a^{\frac{1}{n}}{}^{m} = \left(\sqrt[n]{a}\right)^m$ (HL)

$= \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$

☐ $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ and $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

☐ To rationalize the denominator when it is $\sqrt{a} \pm \sqrt{b}$ you multiply the numerator and denominator of the fraction by $\sqrt{a} \mp \sqrt{b}$.

☐ In standard form you write numbers in the form $a \times 10^n$ where $1 \le a < 10$ and $n \in \mathbb{Z}$.

Chapter 1 test

DP style **Analysis and Approaches SL**

1 $P = \{p \mid p \in \mathbb{Z}, 4 \le p < 10\}$ and $Q = \{q \mid q \in \mathbb{N}, q \le 7\}$, where $U = \mathbb{Z}$

a Write the sets P and Q as lists inside curly brackets $\{\}$.

b Determine which of these statements are true and which are false:

i $8 \in P \cap Q$ **ii** $-2 \in P \cap Q$ **iii** $10 \in P'$ **iv** $0 \in P \cup Q$

c Write the set $P \cap Q$ using set builder notation.

2 Calculate 2.84×37.6, giving your answer to:

a 3 s.f.

b 2 d.p.

3 Find

a $|3 \times (-4)| - |3| \times |-4|$

b $|3 - (-2)| - |3| - |-2|$

c $|(-4)^3| + |-4|^3$

4 Determine which of these statements are true and which are false:

a $2 + (4 - 5) = (2 + 4) - 5$

b $(3 \times 4) \div 2 = 3 \times (4 \div 2)$

c $6 - (2 - 3) = (6 - 2) - 3$

d $24 \div (6 \div 2) = (24 \div 6) \div 2$

5 State which of these numbers is prime. If a number is composite, write it as a product of prime factors.

a 57 **b** 73 **c** 97 **d** 143 **e** 133

6 Simplify each of the following expressions:

a $\frac{3}{4} - \frac{1}{8}$

b $\frac{1}{2} \times \frac{5}{8}$

c $\frac{10}{27} \div \frac{5}{12}$

d $\frac{2}{5} + \frac{1}{10} \times \frac{3}{4}$

7 Simplify each expression.

a $10^3 \times 10^2$ **b** $8^5 \div 8^3$ **c** $\frac{2^4 \times 10^2}{5^3}$

8 Simplify each of the following expressions:

a $\sqrt{18}$

b $\sqrt{24} \div \frac{\sqrt{2}}{2}$

c $\frac{\sqrt{2}}{2} \times \sqrt{24} \times \sqrt{3}$

9 a Write these numbers in standard form:

i 123 580 000 **ii** 0.00127

b Write these numbers in decimal form:

i 2.54×10^5 **ii** 7.68×10^{-2}

DP style **Analysis and Approaches SL**

10 All numbers in this question are written in standard form.

a Given that

$$\frac{1.5 \times 10^p}{2 \times 10^q} = a \times 10^r$$

i find the value of *a*

ii find an expression for *r* in terms of *p* and *q*

b i find an expression for *d* in terms of *b* and *c* given that $b \times 10^6 + c \times 10^7 = d \times 10^7$

ii state an additional constraint that must be satisfied by *b* and *c* and justify your answer.

DP style **Applications and Interpretation SL**

11 Under certain conditions the size of a population of fruit flies can be modelled by the equation

$$N = a2^{bt}$$

where *N* is the size of the population and *t* is the time in weeks from a fixed point.

A population of fruit flies in a large container initially (at $t = 0$) has just 5 fruit flies. After 2 weeks there are 320 fruit flies.

Assuming the equation is a good model for the population:

a find the values of *a* and *b*

b find the size of the population after 3 weeks.

The equation is rewritten in the form $N = ac^t$.

c Write down the value of *c*.

d Find the first week on which the population exceeds 1000.

Higher Level

12 a Find the greatest common factor of 42 and 28.

b Find the least common multiple of 42 and 28.

13 Calculate **a** $2^{-2} \times 2^6 \times 2^{-1}$ **b** $8^{\frac{1}{2}} \times 8^{\frac{1}{6}}$ **c** $\frac{5^4 \times 5^{-2}}{5^2}$

14 Simplify **a** $\frac{6}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2} - 1}$ **c** $\frac{\sqrt{3}}{\sqrt{3} + \sqrt{2}}$

➔ **Reflect**

Did you prefer the MAA or MAI style questions? What do you need more practice in.

Modelling and investigation

DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

Self-management: Managing time and tasks effectively

DP link

This activity is similar in style to the MAI course as it investigates a real-life context.

Building a model of the solar system

	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
Diameter (km)	4879	12 104	12 756	6792	142 984	120 536	51 118	49 528
Distance from the Sun (10^6 km)	57.9	108.2	149.6	227.9	778.6	1433.5	2872.5	4495.1

Elizaveta is going to create a model of the solar system using some balls to represent planets. She has four types of ball available. In order of size with their approximate diameters, these are the balls:

tennis ball: 7 cm	volleyball: 20 cm	football: 22 cm	basketball: 24 cm

She organizes the sizes of the planets into order and groups those that are a similar size.

- Mercury and Mars are the smallest; she represents each of these using a tennis ball.
- Venus and Earth are next in size; she represents each of these using a volleyball.
- Uranus and Neptune are bigger; she represents each of these using a football.
- The largest are Jupiter and Saturn; she represents each of these using a basketball.

To determine whether this arrangement forms an accurate scale model of the true planet sizes, Elizaveta divides the planet diameter (in km) by the diameter of the ball (in cm) used to represent it.

By considering these calculations, comment on whether Elizaveta be able to make a realistic model with these balls.

Elizaveta finds some Styrofoam balls with these diameters: 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 6, 7, 8, 9, 10, 12, 13, 15, 17, 20, 25, 30, 40 (measurements in cm).

Since Mercury is the smallest planet, she chooses the 1 cm ball to represent it.

How many times larger is Venus than Mercury, to 1 dp? Which ball would represent Venus?

How many times larger is Earth than Mercury, to 1 dp? Which ball would represent Earth?

Choose the balls whose sizes would be closest in scale to the other planets.

The distances from the Sun are in units of 10^6 km. If Elizaveta used the same scale to represent these as she did to represent the diameters, Neptune would be 9 km from the Sun in her model! Instead, she uses a scale that is much smaller so that her model will fit in the room.

She begins by placing Mercury 10 cm from the Sun.

How many times further away than Mercury is Venus? Calculate your answer to the nearest cm. How far away from the Sun will Venus be in the model?

On this scale how far would you place Neptune from the Sun? Elizaveta's classroom is 10 m by 10 m. Does she have room for her model?

What would the distances from the Sun be for the 8 planets?

2

Algebra

Learning outcomes

In this chapter you will learn about:

- Manipulation of linear and quadratic algebraic expressions, including factorization and expansion
- Factorizing quadratic functions, including difference of two squares
- Calculating the numerical value of expressions by substitution
- Rearranging formulae
- Addition and subtraction of algebraic fractions (HL only)
- Use of inequalities, $<, \leq, >, \geq$, intervals on the real number line
- Solution of linear equations and inequalities
- Solution of quadratic equations and inequalities with rational coefficients (HL only)
- Solving systems of linear equations in two variables

Key terms

- Expression
- Formula
- Variable
- Substitution
- Simplifying
- Expansion
- Factorization
- Inequality
- Linear
- Quadratic
- Discriminant
- Simultaneous equations
- Elimination

2.1 Manipulation of algebraic expressions

Algebra uses letters in place of numbers in calculations.

For example, let y represent the price of a chocolate bar. If you do not know the price of one chocolate bar, you can still find an **expression** for the price of three chocolate bars:

$$y + y + y = 3 \times y = 3y$$

The letter y is called a **variable** because its value can vary.

When multiplying a number by a letter (e.g. $3 \times y$) or multiplying two letters together (e.g. $a \times b$), you do not need to use the multiplication sign. You should write $3y$ or ab, for example.

The rules of indices also apply to algebraic expressions, so $yy = y^2$, and $y^{\frac{1}{2}} = \sqrt{y}$, for example.

In Algebra a **term** is either a single number or variable, or numbers and variables multiplied together. Terms are separated by + or – signs, or sometimes by divide.

For example, $\underbrace{3x}_{\text{term}} + \underbrace{4}_{\text{term}} = \underbrace{25}_{\text{term}}$.

Simplifying expressions by collecting like terms

Simplifying an algebraic expression involves collecting together terms that are of the same type (sometimes called **like terms**) into a single term. You can collect numbers together; you can collect x's together

DP ready **International-mindedness**

The word algebra comes from the Arabic الجبر (al-jabr), which means 'the reunion of broken parts'.

Note

A combination of letters, numbers and operations (e.g. $+, -, \times, \div$) is an **expression**, e.g. $y + y + y$ or $3y$ are both expressions.

Internal link

You looked at the laws of indices for numbers in section 1.9 of Chapter 1.

and you can collect y's together. You cannot mix terms together that are not alike, so you cannot collect x's and terms with only numbers into a single term.

Example 1

Simplify these expressions:

a $6-2x+3+4x$ **b** $4a+3b-2a+5b$ **c** $x+\frac{1}{x}-3x^2+4x$

a $6-2x+3+4x=\underbrace{6+3}_{\text{numbers}}\ \underbrace{-2x+4x}_{\text{terms in } x}$

$=9+2x$

Collect numbers together and collect terms in x together.

b $4a+3b-2a+5b=\underbrace{4a-2a}_{\text{terms in } a}+\underbrace{3b+5b}_{\text{terms in } b}$

$=2a+8b$

Collect terms in a and collect terms in b.

c $x+\frac{1}{x}-3x^2+4x=(x+4x)+\frac{1}{x}-3x^2$

$=5x+\frac{1}{x}-3x^2$

Terms in $\frac{1}{x}$ and x^2 are different and cannot be mixed with those in x.

Exercise 2.1a

Simplify each of the expressions in questions 1 – 4.

1 $8-5x+2-3x$ **2** $3a+2b-a+3b$ **3** $2a+3a^2+a-a^2$ **4** $\frac{1}{x}-\frac{2}{y}+\frac{3}{x}$

Expanding single brackets

When an expression contains brackets, you should multiply the term contained outside the bracket by the term contained inside the bracket, e.g. $a(b+c)=a\times(b+c)$.

For example, suppose $a=2$, $b=4$, $c=5$. Then

$$a(b+c)=2(4+5)=2\times9=18 \text{ and}$$

$$ab+ac=2\times4+2\times5=8+10=18$$

Applying the distributive law is known as **expanding the bracket**. You can expand brackets which contain both numbers and variables.

Key point

The **distributive law** states that $a(b+c)=ab+ac$.

Example 2

Expand these expressions.

a $2(x+3)$ **b** $2(x-3)$ **c** $a(b-a)$

a $2(x+3)=2x+2\times3$

$=2x+6$

Multiply each term inside the bracket by 2.

Simplify the term 2×3. You cannot simplify $2x$ any further, so you leave it like this.

b $2(x-3)=2x+2\times(-3)$

$=2x-6$

When multiplying out the second term in the bracket, take care to remember the negative sign.

c $a(b-a)=ab+a(-a)$

$=ab-a^2$

Multiplying a by $(-a)$ gives $-a^2$

Internal link

You learned to simplify expressions by collecting like terms in section 2.1a.

After expanding brackets, you may be left with an expression containing like terms that can be simplified.

Example 3

Expand the brackets and simplify by collecting like terms.

a $2(x-3)+3(4-x)$ **b** $p(p+q)-q(p+q)$

a $2(x-3)+3(4-x) = 2x-6+12-3x$	Expand each of the brackets.
$= \underbrace{12-6}_{\text{numbers}} + \underbrace{2x-3x}_{\text{terms in } x}$	Collect like terms.
$= 6-x$	Simplify.
b $p(p+q)-q(p+q) = p^2+pq-pq-q^2$ $= p^2-q^2$	Expand the brackets. Remember that $pq = qp$.

Exercise 2.1b

1 Expand these expressions.

a $3(a-b)$ **b** $b(2a+c)$ **c** $x(x+4)$ **d** $xy(y-x)$

2 Expand these expressions.

a $2a(a+b+c)$ **b** $(ab-cd)bc$

3 **a** $3(x-2)+4(x+4)$ **b** $2(2a-3)-3(a-4)$

DP link

You will expand and factorize algebraic expressions in both MAA and MAI DP maths.

Factorizing single brackets

The opposite process to expanding a bracket is known as **factorization**.

$$\underset{\xleftarrow{\text{Factorizing}}}{\overset{\xrightarrow{\text{Expanding}}}{5(x+4)=5x+20}}$$

Internal link

In section 1.7 you learned how to find the greatest common factor.

The factor that you should put outside the bracket is the greatest common factor of both terms.

Example 4

Factorize these expressions.

a $3x+18$ **b** $20x+10$ **c** $8a-12$

a $3x+18 = 3(x+6)$	Find the greatest common factor of both terms. Put this outside the bracket and then divide each term by this factor.
b $20x+10 = 10(2x+1)$	
c $8a-12 = 4(2a-3)$	

Sometimes, the greatest common factor of the terms in an expression might contain both a number and a variable. For example, the greatest common factor of $6x$ and $9x^2$ is $3x$. Hence, $6x+9x^2 = 3x(2+3x)$.

Example 5

Factorize these expressions.

a $6x^2 - 3x$ **b** $6a^2 + 4ab - 10a^3$

a $6x^2 - 3x = 3x(2x - 1)$

b $6a^2 + 4ab - 10a^3 = 2a(3a + 2b - 5a^2)$

Find the greatest common factor of both terms. Put this outside the bracket and then divide each term by this factor.

Exercise 2.1c

1 Factorize these expressions.

a $5x + 15$ **b** $8x - 4$ **c** $12a + 9$

2 Factorize these expressions.

a $4p + 6q$ **b** $abc - bd$ **c** $2x^2 - 4x$ **d** $x^2y + 3xy^3$

3 Factorize these expressions.

a $6a^2b + 9ab^2$ **b** $x^2yz^3 + x^4z^2$

2.2 Calculating the numerical value by substitution

Key point

When you know the values of the **variables** in an expression, you can calculate the value of the expression by **substitution**.

Example 6

When $a = 2$, $b = -3$ and $c = 5$, calculate the values of these expressions.

a $2a^2 + b$ **b** $\frac{c - b}{a}$ **c** $b(a + c)$

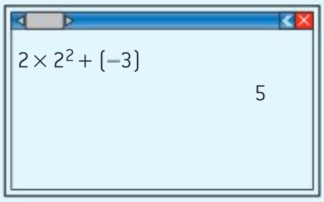
a
$$\begin{aligned} 2a^2 + b &= 2 \times 2^2 + (-3) \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

b
$$\begin{aligned} \frac{c - b}{a} &= \frac{5 - (-3)}{2} \\ &= \frac{8}{2} \\ &= 4 \end{aligned}$$

c
$$\begin{aligned} b(a + c) &= -3(2 + 5) \\ &= -3 \times 7 \\ &= -21 \end{aligned}$$

Substitute for a, b and c in each expression.

If you are using a calculator, enter the whole expression, just as you would write it on paper, rather than breaking it down, e.g. for part **a**



$2 \times 2^2 + (-3)$

5

When entering a negative number into your calculator, you should place it in parentheses.

Key point

An equation states that two mathematical expressions are equal to one another using an '=' sign.

Examples of equations are:

$$2 + 3 = 15 - 10$$

$$3 \times 4 = 12$$

$$x^2 + y = 2x$$

Key point

A formula is a special type of equation that gives a rule using symbols for variables.

Internal link

You will study solutions to linear equations in 2.5.

For example, a formula for v is $v = u + at$

Substitution into a formula or equation works in exactly the same way as into an expression. Replace whichever variables you are given the value of with numbers, and then calculate the values of any remaining unknown variables.

Example 7

The formula $v = u + at$ gives the final speed v of a car, where u is initial velocity, a is acceleration and t is time. Find the value of v when:

a $u = 10\text{ m s}^{-1}$, $a = 5\text{ m s}^{-2}$ and $t = 20\text{ s}$

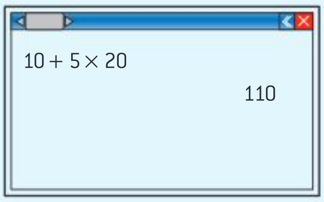
b $u = 60\text{ ms}^{-1}$, $a = -4\text{ ms}^{-2}$ and $t = 12\text{ s}$

a $v = u + at$

$= 10 + 5 \times 20$

$= 110\text{ m s}^{-1}$

Substitute for u, a and t in the given formula.



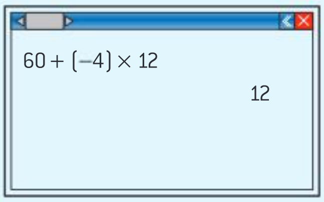
```
10 + 5 × 20
                 110
```

b $v = u + at$

$= 60 + (-4) \times 12$

$= 12\text{ m s}^{-1}$

Substitute for u, a and t in the given formula.



```
60 + (−4) × 12
                  12
```

Internal link

Remember to consider the correct order of operations in your calculations (which you learned in section **1.5**) if you are not using a GDC.

DP link

The unit for speed used in the DP course is m s^{-1}. This means 'metres per second', or 'm/s'. Can you explain why there is a power of −1 on the 's'?

Exercise 2.2

1 If $p = 2$, $q = -6$ and $r = 5$, find the values of:

a $p(q + r)$ **b** $\dfrac{p^2 + q}{p - r}$ **c** $\sqrt{\dfrac{1}{2}(p - q)}$

2 The formula $s = ut + \frac{1}{2}at^2$ gives the distance travelled by a train, where u is its initial velocity, a is its acceleration and t is time. If $u = 15\text{ m s}^{-1}$, $a = 8\text{ m s}^{-2}$ and $t = 10\text{ s}$, find s.

2.3 Addition and subtraction of algebraic fractions

In this section you will learn how you can apply the rules of adding and subtracting fractions to **algebraic fractions**. An example of an algebraic fraction is $\frac{3}{4x}$ or $\frac{1}{2x+3}$.

Recall that to add (or subtract) two numerical fractions you first had to find a common denominator and then add the numerators. For example, $\frac{1}{3}+\frac{1}{2}=\frac{2+3}{6}=\frac{5}{6}$.

When adding algebraic fractions, you find a common denominator as well. When the denominators are different and have no common factors, the common denominator is the product of the denominators. If the denominators have a common factor, then the common denominator is their least common multiple.

Internal link

In section **1.8** you learned how to add and subtract fractions involving only numbers.

Key point

In **algebraic fractions**, either the numerator or the denominator (or both) contains a variable.

Internal link

You learned about least common multiples in section **1.7**.

Example 8

a $\frac{3}{5x}+\frac{2}{3}$ **b** $\frac{1}{x+1}-\frac{2}{2x+3}$ **c** $\frac{1}{(x-1)^2}-\frac{1}{x-1}$

a

$$\frac{3}{5x}+\frac{2}{3}=\frac{3}{5x}\times\frac{3}{3}+\frac{2}{3}\times\frac{5x}{5x}$$
$$=\frac{9}{15x}+\frac{10x}{15x}$$
$$=\frac{9+10x}{15x}$$

The common denominator is $5x\times 3=15x$.

Multiply first term by $\frac{3}{3}$ and second term by $\frac{5x}{5x}$

b

$$\frac{1}{x+1}-\frac{2}{2x+3}=\frac{1(2x+3)-2(x+1)}{(x+1)(2x+3)}$$
$$=\frac{2x+3-2x-2}{(x+1)(2x+3)}$$
$$=\frac{1}{(x+1)(2x+3)}$$

The common denominator is $(x+1)(2x+3)$

There is no need to expand the brackets in the denominator.

c

$$\frac{1}{(x-1)^2}-\frac{1}{x-1}=\frac{1}{(x-1)^2}-\frac{(x-1)}{(x-1)^2}$$
$$=\frac{1-(x-1)}{(x-1)^2}$$
$$=\frac{1-x+1}{(x-1)^2}$$
$$=\frac{2-x}{(x-1)^2}$$

Since $x-1$ is a factor of $(x-1)^2$ the common denominator is the least common multiple, which is $(x-1)^2$ in this case.

DP link

DP students who study MAA at HL learn how to reverse this process: splitting a rational algebraic expression into a sum of two algebraic fractions, known as partial fractions.

Exercise 2.3a

1 Find the sum of:

a $\frac{2}{3x} - \frac{1}{x}$ **b** $\frac{3}{4a} + \frac{2a}{3}$ **c** $\frac{1}{6a^2} + \frac{3}{4a}$ **d** $\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3}$

2 Find the sum of:

a $\frac{1}{x+1} + \frac{1}{x+2}$ **b** $\frac{x}{x-1} + \frac{1}{2x}$

c $\frac{x}{2x-3} - \frac{1}{x+2}$ **d** $\frac{2x+1}{2x-1} + \frac{2}{x}$

Command term

Show that means you should obtain the required result (possibly using information given) without the formality of proof. 'Show that' questions do not generally require the use of a calculator.

State means you should give a specific name, value or other brief answer without explanation or calculation.

Verify means you should provide evidence that validates the result. In this case, you need to check each of the fraction sums $\frac{1}{3}+\frac{1}{6}, \frac{1}{4}+\frac{1}{12}, \ldots$ can be written in the form $\frac{1}{m+1}+\frac{1}{m(m+1)}$.

Prove means that you should use a sequence of logical steps to obtain the required result in a formal way.

DP style Analysis and Approaches HL

3 **a** **Show that** $\frac{1}{(1+1)} + \frac{1}{(1 \times 2)} = 1$ and $\frac{1}{(2+1)} + \frac{1}{(2 \times 3)} = \frac{1}{2}$

Find $\frac{1}{(3+1)} + \frac{1}{(4 \times 5)}$

State what you notice about the sums.

b Use the pattern above to write down the next pair of fractions in the sequence starting:

$\frac{1}{2}+\frac{1}{2}, \frac{1}{3}+\frac{1}{6}, \frac{1}{4}+\frac{1}{12}, \ldots$

State whether the sum of each term follows the same pattern, or a different one. Write down an expression for the *n*th term.

c **Verify** that the sums $\frac{1}{2}+\frac{1}{2}, \frac{1}{3}+\frac{1}{6}, \frac{1}{4}+\frac{1}{12}$, and the sum of the next pair of fractions can be generalised by the expression $\frac{1}{m+1}+\frac{1}{m(m+1)}$.

d **Prove** that $\frac{1}{m+1}+\frac{1}{m(m+1)}$ is equal to the general term for the sum of each pair of fractions, which you found in part **b**.

Internal link

In section **2.1** you learned how to expand and factorize expressions.

You can also expand and factorize expressions which involve algebraic fractions.

Example 9

Expand and simplify $\frac{1}{6a^2}(3a+2)$.

$$\frac{1}{6a^2}(3a+2) = \frac{3a}{6a^2} + \frac{2}{6a^2}$$

$$= \frac{1}{2a} + \frac{1}{3a^2}$$

Here, you can consider numbers that are not fractions as being over denominator 1, that is

$$\frac{1}{6a^2}\left(\frac{3a}{1}+\frac{2}{1}\right) = \frac{1}{6a^2} \times \frac{3a}{1} + \frac{1}{6a^2} \times \frac{2}{1}$$

There are no like terms, but you can simplify the first fraction by dividing through by $3a$ top and bottom, and the second by dividing through by 2.

Example 10

Factorize $\frac{3x}{4}+\frac{5x^2}{6}$

$$\frac{3x}{4}+\frac{5x^2}{6}=\frac{9x}{12}+\frac{10x^2}{12}$$

$$=\frac{x}{12}(9+10x)$$

To factorize an expression containing algebraic fractions, first write both terms over a common denominator (see section 1.8 for a recap of how to do this).

You can then find the greatest common factor of both terms, and factorize.

Exercise 2.3b

1 Expand these expressions.

a $\frac{1}{2}(4x-6y)$ **b** $\frac{3}{4x}(6x^2-12x)$

2 Factorize these expressions.

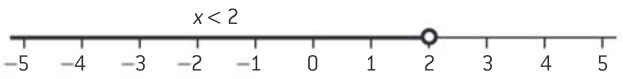
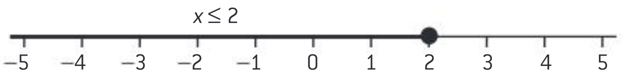
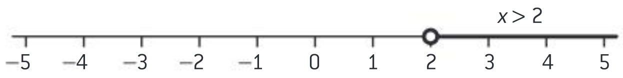
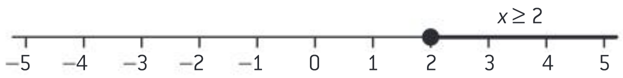
a $\frac{x^2}{6}+\frac{x}{4}$ **b** $ut+\frac{1}{2}at^2$

2.4 Use of inequalities

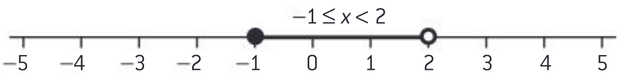
There are four inequality symbols: <, ≤, > and ≥.

Internal link

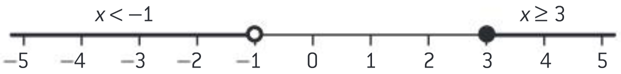
You briefly studied inequalities such as $x \le 10$ in section 1.2 on Sets.

Symbol	Meaning	Diagram	Explanation
<	'less than'		Open circle means 2 **is not** included.
≤	'less than or equal to'		Closed circle means 2 **is** included.
>	'greater than'		Open circle means 2 **is not** included.
≥	'greater than or equal to'		Closed circle means 2 **is** included.

You can also use inequalities to represent a set of numbers in between two values. For example, $-1 \le x < 2$ means the numbers between –1 and 2, where –1 is included but 2 is not.



You also need to be able to write inequalities that are *not* between certain values. Here, you have to write them in two parts. For example, $x < -1$ and $x \ge 3$ means numbers not between –1 and 2.



What you include in the inequality depends on which numbers are in the universal set; it may be the integers, the rational numbers, or the real numbers.

Internal link

You studied sets of numbers in section 1.2.

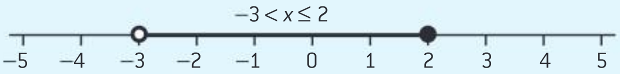
Example 11

a Determine whether the inequality $\frac{5}{6} \leq 0.83$ is true or false.

b **Draw** a diagram to show the region $-3 < x \leq 2$.

a $\frac{5}{6} = 0.8333\ldots$, $0.8333\ldots > 0.83$ hence $\frac{5}{6} \leq 0.83$ is false.

b



To compare fractions and decimals, write the fraction as a decimal (or vice versa).

Command term

Draw means you should represent by means of a labelled, accurate diagram, using a pencil. Diagrams should be drawn to scale.

You can manipulate inequalities by reversing them. For example, the inequality $3 < 6$ is equivalent to $6 > 3$. You reverse the inequality and use the opposite sign. Similarly, $-2 \geq x$ is equivalent to $x \leq -2$.

Exercise 2.4

1 Complete the statement $0.6 \square \frac{11}{17}$ using ≥ or ≤.

2 If $a \in \mathbb{Z}$, list all the values of a for which $2 < a \leq 9$.

3 Determine which of these statements are true and which are false:

a $2 < -3$ **b** $3 \div 5 \leq 0.6$ **c** $\frac{14}{23} \leq 0.6$ **d** $2 \times 5 - 6 \geq 10$

4 Draw diagrams to show the regions represented by these inequalities:

a $x < -1$ **b** $2 \leq x$ **c** $x \geq 1$ **d** $1 < x \leq 4$

Internal link

Recall from section 2.2: an equation states that two mathematical expressions are equal to one another using an '=' sign.

2.5 Solutions to linear equations

Manipulating an equation

If you do the same thing on both sides of an equation, the left-hand side and right-hand side remain equal. For example, suppose you have the equation $2 + 4 = 6$.

Adding 2 to both sides of the equation:

$$2 + 4 + 2 = 6 + 2$$

The equation still holds, because both sides equal 8.

Now multiplying both sides by 3:

$$3(2 + 4 + 2) = 3(6 + 2)$$

The equation still holds, because both sides equal 24 (try expanding the brackets and check!)

The same is true if you subtract or divide both sides of an equation by the same number.

This is called **manipulating** the equation.

Solving equations

Algebraic equations contain a variable, such as x, which is an unknown value. To **solve** an algebraic equation you need to find the numerical value of x which makes both sides of the equation the same.

You solve an equation for x by manipulating the equation to **make x the subject**.

Command term

Solve means you should obtain the answer using algebraic methods.

Example 12

a **Solve** the equation $2x+3=11$ and **verify** the result.

b Solve the equation $3(2x-4)=18$.

a $2x+3-3=11-3$ $2x=8$	Subtract 3 from both sides of the equation
$\frac{2x}{2}=\frac{8}{2}$ $x=4$	Divide both sides of the equation by 2 You are left with an equation where x is on its own on the left-hand side, with a number on the right-hand side. You have **solved** the equation.
Checking: $2\times 4+3=8+3$ $=11$	Substitute the solution back in the equation to verify the result. (You learned how to substitute in section 2.2).
b $3(2x-4)=18$ $(\div 3)$ $2x-4=6$ $(+4)$ $2x=10$ $(\div 2)$ $x=5$	Divide both sides of the equation by 3. Add 4 to both sides of the equation. Divide both sides of the equation by 2.
Alternatively: $3(2x-4)=18$ $6x-12=18$	You could alternatively expand the bracket first.
$(+12)$ $6x=30$ $(\div 6)$ $x=5$	Add 12 to both sides of the equation. Divide both sides of the equation by 6.

In some equations, the unknown, x, occurs in more than one place; for example $4x-5=x+4$. To solve these, you must first collect all the terms containing x on one side.

Example 13

Solve the equation $4x-5=x+4$

$4x-5=x+4$ $3x-5=4$ $3x=9$ $x=3$	Subtract x from both sides of the equation. Add 5 to both sides of the equation. Divide both sides of the equation by 3.

Other equations involving fractions and expressions in brackets can be solved by algebraic manipulation.

Example 14

a Solve $\frac{x}{2}+6=10$

b Solve $3(x-2)-4(2-x)-x=x+6$

c Solve $\frac{x+1}{2}=\frac{x-2}{3}$

a	$\frac{x}{2}+6=10$	Subtract 6 from both sides.
	$\frac{x}{2}=4$	Multiply both sides by 2.
	$2\left(\frac{x}{2}\right)=2\times 4$	
	$x=8$	
b	$3(x-2)-4(2-x)-x=x+6$	
	$3x-6-8+4x-x=x+6$	Expand.
	$6x-14=x+6$	Simplify.
	$5x-14=6$	Subtract x from both sides.
	$5x=20$	Add 14 to both sides.
	$x=4$	Divide both sides of the equation by 5.
c	$\frac{x+1}{2}=\frac{x-2}{3}$	
	$6\left(\frac{x+1}{2}\right)=6\left(\frac{x-2}{3}\right)$	Multiply both sides of the equation by 6 (equivalent to multiplying by 3 and 2).
	$3(x+1)=2(x-2)$	
	$3x+3=2x-4$	
	$x+3=-4$	Subtract $2x$ from both sides.
	$x=-7$	Subtract 3 from both sides.

Exercise 2.5a

1 Solve these equations.

a $4x-7=21$ **b** $8=2x-10$ **c** $12-3x=6$ **d** $\frac{x}{4}+3=7$

2 Solve these equations.

a $5x-2=x+6$ **b** $2-x=3x-6$ **c** $3x-5=4-3x$ **d** $\frac{x+1}{10}=\frac{x-1}{5}$

3 Solve these equations.

a $x+2(x-1)=7$ **b** $2(x+3)-3(2-x)=4(x-5)$

c $3(2x-3)-(x+1)=4(1-x)+4$ **d** $\frac{3}{4}(2x+3)+\frac{1}{2}(x-4)=0$

Forming and solving linear equations

Equations are sometimes in the form of problems and to solve them you need to write them using algebra. Identify the unknown and call it x, or another letter, then write the equation algebraically.

Example 15

a The sum of three consecutive numbers is 342. Find the numbers.

b The width of a rectangle is twice its length. If the perimeter is 18 cm, find the length and width of the rectangle.

a Let one of the numbers be x. The other numbers are $x+1$ and $x+2$.	**Consecutive** means following each other in order.
$x+x+1+x+2=342$ $3x+3=342$	Write the information in the problem as an equation.
$3x=339$	Subtract 3 from both sides of the equation.
$x=113$	Divide both sides of the equation by 3.
The numbers are 113, 114 and 115	
b Let the length of the rectangle be x	
The width is twice the length so $2x$	
$2(x+2x)=18$	The perimeter is 2(length + width).
$2(3x)=18$	Simplify and expand.
$6x=18$	Divide both sides of the equation by 6.
$x=3$	
The length is 3 cm and the width 6 cm.	

Exercise 2.6b

1 Rahul is 26 years younger than his mother. Their total age is four times his age. How old is he?

2 The sum of three numbers is 33. The second number is three times the first and the third is 2 less than the second. What are the numbers.

DP ready **Theory of knowledge**

Remember that an equation is a statement with an = sign, stating that two expressions are equal in value. You have been looking at equations that have a solution. Not all equations can be solved however.

Consider $4x+3=4x-5$. This equation has *no* solution, there is no value of x that will make the expressions on either side of the = sign equal.

Now consider $4(x+2)=4x+8$. This equation has *infinitely many* solutions. For any value of x the expressions on either side of the = sign are equal. This is known as an **identity** and the sign ≡ is used in place of = to show this.

Once they have been discovered, identities are often very useful to mathematicians as they enable mathematical expressions to be written often in a simpler way. An example of an identity is the difference of two squares $a^2-b^2\equiv(a+b)(a-b)$.

Was the difference of two squares result discovered (in the way that Marie and Pierre Curie discovered radium) or was it invented (in the way that Michelin invented the tire)?

Rearranging and substitution

In section 2.2 you used the formula $v=u+at$ to find v when you knew the values of u, a and t. What could you do if you knew the values of v, u and a and wanted to find the value of t?

Using what you learned in 2.6a, you can manipulate the equation to make t the subject, and then substitute the other values in.

Example 16

a Use the equation $v=u+at$ to find u if $v = 8\text{ m s}^{-1}$, $a = 2\text{ m s}^{-2}$ and $t = 3$ s.

b Use the equation $v=u+at$ to find t if $u = 6\text{ m s}^{-1}$, $v = 2\text{ m s}^{-1}$ and $a = -2\text{ m s}^{-2}$

a $v = u + at$	Since u is going to be the subject, keep it on the side of the formula where it will be positive.
$v - at = u$	Subtract at from both sides.
$u = v - at$	u is the subject, so rewrite the formula so that it is on the left-hand side.
$u = 8 - 2 \times 3$ $u = 2\ \text{m s}^{-1}$	Substitute values for v, a and t.
b $v = u + at$	Isolate the term containing t on the side of the formula where it is positive.
$v - u = at$	Subtract u from both sides.
$\frac{v-u}{a} = t$	Divide both sides by a.
$t = \frac{v-u}{a}$	Rewrite the formula so that t is on the left-hand side.
$t = \frac{2-6}{-2}$ $= 2\ \text{s}$	Substitute values for v, a and u.

Note

You could substitute for the known values before you manipulate the equation if you prefer.

Exercise 2.5c

1 Rearrange these formulae to make x the subject.

a $y = 2x + 1$ **b** $y = 4(x - 5)$ **c** $y = 5 - 2x$ **d** $y = \frac{3}{x}$

2 The total surface area A of a cone with radius r and slant height l is given by the formula $A = \pi r(r + l)$. Rearrange the formula to make the slant height l the subject.

3 Ohm's law states that the current I through a conductor with voltage V and resistance R is given by $I = \frac{V}{R}$. Rearrange the formula to give:

- **a** voltage in terms of current and resistance
- **b** resistance in terms of voltage and current.

4 The volume V of a cuboid with length l, width w and height h is given by $V = lwh$.

Rearrange the formula so that it gives h in terms of V, l and w.

If $V = 4.32\ \text{cm}^3$, $l = 2\ \text{cm}$ and $w = 5.4\ \text{cm}$, calculate h.

DP style **Applications and Interpretation HL**

5 The total resistance R_t of two resistances R_1 and R_2 in a parallel circuit is given by the formula $\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2}$.

- **a** Rearrange the formula to make R_t the subject.
- **b** Hence find R_t when $R_1 = 1.2$ and $R_2 = 2.5$

Simplifying expressions involving roots

In this section you have used the operations of addition, subtraction, multiplication and division to rearrange algebraic formulae. Some formulae also contain squared or square root terms.

You can also transform formulae by squaring or taking the square root of both sides so that if $x = y$, then $x^2 = y^2$ and $\sqrt{x} = \sqrt{y}$.

Example 17

a Make l the subject of the formula $t = 2\pi\sqrt{\frac{l}{g}}$.

b Rearrange the formula $F = G\frac{m_1m_2}{r^2}$ to give r in terms of the other variables.

a $t = 2\pi\sqrt{\frac{l}{g}}$	
$\frac{t}{2\pi} = \sqrt{\frac{l}{g}}$	Make the square root the subject of the equation.
$\frac{t^2}{4\pi^2} = \frac{l}{g}$	Square both sides. Make make sure that each term is squared.
$\frac{gt^2}{4\pi^2} = l$	Multiply both sides by g.
Then $l = \frac{gt^2}{4\pi^2}$	Rewrite the formula so that l is on the left-hand side.
b $F = G\frac{m_1m_2}{r^2}$	
$Fr^2 = Gm_1m_2$	Multiply both sides by r^2.
$r^2 = G\frac{m_1m_2}{F}$	Divide both sides by F.
$r = \sqrt{G\frac{m_1m_2}{F}}$	Take square root of both sides.

Exercise 2.5d

1 Make r the subject of $A = 4\pi r^2$.

2 Rearrange $a = \sqrt{b - qt}$ to make t the subject.

3 The length of the diagonal of a right-angled triangle is given by the formula $a = \sqrt{b^2 + c^2}$. Rearrange the formula to give b in terms of a and c.

Note

In general, when taking the square root you will have two answers, one positive and one negative. However, in part **b** of the worked example above, r stands for radius which can only be positive, so only one answer is possible.

2.6 Solutions to linear inequalities

A linear inequality is much like a linear equation but instead of writing it with an = sign, one of the four inequality symbols: <, ≤, > and ≥ is used. For example, $2x + 5 \leq 11$ means '$2x + 5$ is less than or equal to 11'.

To solve an inequality, you manipulate it in a similar way to an equation. Inequalities, however, are different to equations in some important ways.

Investigation 2.1

a Consider the inequality $4 \leq 8$.

- **i** Add 3 to both sides of the inequality. Is it still true?
- **ii** Add –2 to both sides of the inequality. Is it still true?
- **iii** Subtract 1 from both sides of the inequality. Is it still true?
- **iv** Subtract –4 from both sides of the inequality. Is it still true?

v Multiply both sides of the inequality by 2. Is it still true?
vi Multiply both sides of the inequality by –3. Is it still true?
vii Divide both sides of the inequality by 4. Is it still true?
viii Divide both sides of the inequality by –2. Is it still true?

b From your results, **suggest** what happens when you add to, subtract from, multiply by or divide by either a positive or negative number.

c What happens when you add, subtract, multiply or divide by zero?

d What happens with other inequalities? Are the rules the same as they are for $\leq$?

Command term

Suggest means you should propose a solution, hypothesis or other possible answer.

Key point

You can solve inequalities by algebraic manipulation:

- adding the same number to, or subtracting it from, both sides does not change the inequality
- multiplying or dividing both sides by the same *positive* number does not change the inequality
- multiplying or dividing both sides by the same *negative* number reverses the direction of the inequality.

Example 18

Solve the inequalities.

a $2x + 5 \leq 11$ **b** $8 - 2x < 10$

a $2x + 5 \leq 11$ $2x \leq 6$ $x \leq 3$	Subtract 5 from both sides of the inequality. Divide both sides of the inequality by 2.
b $8 - 2x < 10$ $-2x < 2$ $x > -1$	Subtract 8 from both sides of the inequality. Divide both sides by –2. Note that this reverses the inequality.
or $8 - 2x < 10$ $8 < 10 + 2x$ $-2 < 2x$ $-1 < x$ $x > -1$	Add $2x$ to both sides of the inequality. Subtract 10 from both sides of the inequality. Divide by 2. Note: Here, the inequality has not been reversed (as it would be if dividing by a negative number). We merely write the inequality the other way around, with x coming first.

Exercise 2.6

1 Solve the inequalities.

a $3x - 8 \leq 13$ **b** $5 \geq 2x - 3$

c $x - 2(2x - 1) \geq 0$ **d** $4 - \frac{2x}{3} < 8$

2.7 Factorizing quadratics

When you expand double brackets, you get a quadratic expression in x. This is an expression of the form $ax^2 + bx + c$ where the highest power of x is 2.

Internal link

You learned how to expand single brackets in section 2.1b.

Example 19

Expand these expressions.

a $(x+2)(x+3)$ **b** $(x-2)(x-4)$ **c** $(a-3)(a+3)$

a $(x+2)(x+3) = x(x+3)+2(x+3)$ $= x^2 + 3x + 2x + 6$ $= x^2 + 5x + 6$	Split the first bracket. Expand both single brackets. Collect like terms.
b $(x-2)(x-4) = x(x-4)-2(x-4)$ $= x^2 - 4x - 2x + 8$ $= x^2 - 6x + 8$	Take care with negative signs.
c $(a+3)(a-3) = a(a-3)+3(a-3)$ $= a^2 - 3a + 3a - 9$ $= a^2 - 9$	

Internal link

This result is the difference of two squares that you used in rationalizing the denominator in the HL extension to section 1.10.

You can also factorize some quadratic expressions $ax^2 + bx + c$. Below you will learn how to do this for different types of quadratic expression.

Case 1: $c = 0$.

If c is zero, then there are two terms with a common factor of x. You should factorize x and any numerical factors from both terms.

Example 20

Factorize $4x^2 - 6x$.

$4x^2 - 6x = 2x(2x-3)$	In this expression, x is a common factor as is the numerical factor 2. So, you take $2x$ out as a factor.

Higher Level

Case 2: $b = 0$.

If b is zero, then, in some situations, you can factorize the quadratic expression as the difference of two squares.

Consider $(a+b)(a-b)$. Expand the brackets to obtain $a^2 - ab + ab - b^2$. Simplifying, this is $a^2 - b^2$. You can use this result to factorize any expression that can be written as the difference of two squares.

If you can write the quadratic as $(px)^2 - q^2$ then the factors are $(px+q)(px-q)$.

Example 21

Factorize $9x^2 - 16$.

$9x^2 - 16 = (3x+4)(3x-4)$	Since $9x^2 = (3x)^2$ and $16 = 4^2$

Case 3: $a = 1$.

Consider $(x+3)(x+4)$. Multiplying out the brackets, you get $x^2+7x+12$, where $7 = 3+4$ and $12 = 3 \times 4$. So, to factorize $x^2+7x+12$, you would look for two numbers that multiply to make 12 and add to make 7. You can generalize this process. To factorize x^2+bx+c, look for two numbers that multiply to make c and add to make b.

Example 22

Factorize:

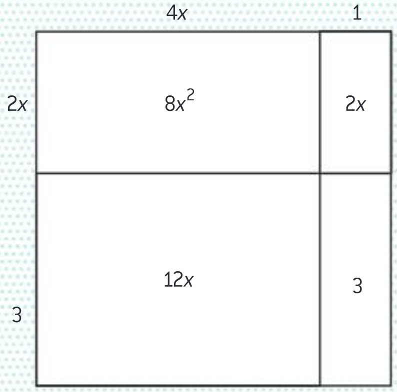
a $x^2+7x+10$ **b** $x^2+4x-12$ **c** x^2-4x+4

a $x^2+7x+10=(x+2)(x+5)$	Find two numbers that multiply to make 10. For example, 1 and 10, –1 and –10, –2 and –5, or 2 and 5. Since 2 + 5 = 7, the factors are $(x+2)$ and $(x+5)$.
b $x^2+4x-12=(x+6)(x-2)$	The factors of –12 are –3 and 4, 3 and –4, –6 and 2, 6 and –2, –1 and 12 or 1 and –12. The pair that add up to 4 are 6 and –2, so the factors are $(x+6)$ and $(x-2)$.
c $x^2-4x+4=(x-2)(x-2)$ $=(x-2)^2$	The two numbers that multiply to make 4 are 1 and 4, –1 and –4, 2 and 2 or –2 and –2. The pair that add up to –4 are –2 and –2 so the factors are $(x-2)$ and $(x-2)$.

Higher Level

Case 4: $a > 1$.

Consider $(2x+3)(4x+1)$.



$(2x+3)(4x+1) = 8x^2+12x+2x+3 = 8x^2+14x+3$

In order to factorize $8x^2+14x+3$, you first want to split $14x$ into $12x$ and $2x$.

To do this, multiply $8 \times 3 = 24$. Then see if there are any factors of 24 which add up to 14.

$12 \times 2 = 24$ and $12 + 2 = 14$, so 12 and 2 are the numbers you need.

Split the middle term: $8x^2+14x+3 = 8x^2+12x+2x+3$

Factorize the first two and last two terms separately:

$$8x^2+12x+2x+3 = 4x(2x+3)+1(2x+3)$$

Since both brackets are the same, factor the bracket out:

$$4x(2x+3)+1(2x+3) = (2x+3)(4x+1)$$

Note

Try factorizing $3x^2-5x-2$ but this time reverse the order of the two middle terms (x and $-6x$) so you have $3x^2-6x+x-2$. Does this make a difference to the final result?

Example 23

Factorize $3x^2 - 5x - 2$.

$3 \times (-2) = -6$	Multiply a by c and find its factors.
Factors of –6 include: 1 and –6, or –1 and 6 Now 1 + (–6) = –5, so we need 1 and –6	Find two numbers that add up to –5 and multiply to make $3 \times (-2) = -6$.
$3x^2 - 5x - 2 = 3x^2 + x - 6x - 2$	Split the middle term.
$= x(3x+1) - 2(3x+1)$	Factorize both pairs of terms.
$= (3x+1)(x-2)$	Factor out $(3x+1)$.

Exercise 2.7

1 Expand:

a $(x+1)(x+3)$ **b** $(x-4)(x+2)$ **c** $(x+2)(x-5)$ **d** $(x-3)(x-5)$

2 Factorize:

a $3x - 6x^2$ **b** $2y^2 + 3y$ **c** $4x^2 - 9$ **d** $4 - x^2$

3 Factorize:

a $x^2 + 5x + 6$ **b** $x^2 - 6x + 8$ **c** $x^2 + 2x - 3$ **d** $x^2 - x - 20$

Higher Level

4 Expand:

a $(3x-2)(3x+2)$ **b** $(2x+1)(3x-4)$

5 Factorize:

a $3x^2 + 8x + 4$ **b** $2x^2 + x - 21$ **c** $2x^2 - x - 15$ **d** $6x^2 + 7x - 3$

Higher Level

2.8 Solutions to quadratic equations and inequalities

You can form a quadratic equation by letting a quadratic expression be equal to zero. If you factorize the quadratic expression then you will have the product of two factors equal to zero.

If two numbers multiply to make zero, that is $a \times b = 0$, then what can you say about a and b? Either a or b must be zero. Either $0 \times b = 0$ or $a \times 0 = 0$; no other combination of numbers will give that result.

Look at the quadratic equation $x^2 - 5x + 6 = 0$.

Factorizing gives $(x-3)(x-2) = 0$.

So either $x - 3 = 0$ and $x = 3$, or $x - 2 = 0$ and $x = 2$.

Generally, a quadratic equation has two solutions (sometimes called roots).

Example 24

Solve these quadratic equations by factorizing.

a $2x^2 - 4x = 0$

b $4x^2 - 16 = 0$

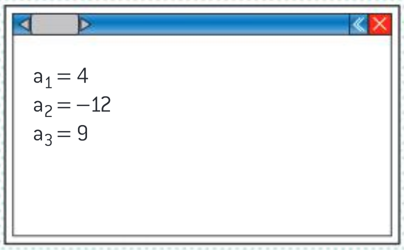
c $x^2 + 2x - 15 = 0$

d $4x^2 - 12x + 9 = 0$

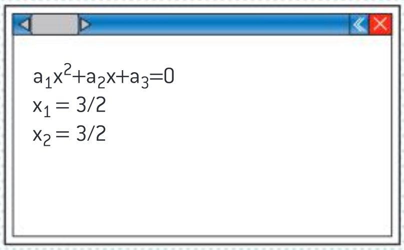
e The length of a rectangle is 2 cm longer than its width. If the area is 15 cm², find the dimensions of the rectangle.

a $2x^2 - 4x = 0$ $2x(x-2) = 0$ $x = 0$ or 2	Factorize the quadratic. Let the bracket equal zero, and let $2x$ equal zero.
b $4x^2 - 16 = 0$ $(2x-4)(2x+4) = 0$ $x = 2$ or -2	Factorize using the difference of two squares. Let each bracket equal zero.
c $x^2 + 2x - 15 = 0$ $(x-3)(x+5) = 0$ $x = 3$ or $x = -5$	Factorize the quadratic. Let each bracket equal zero.
d $4x^2 - 12x + 9 = 0$ $4x^2 - 6x - 6x + 9 = 0$ $2x(2x-3) - 3(2x-3) = 0$ $(2x-3)(2x-3) = 0$ $(2x-3)^2 = 0$ $x = \frac{3}{2}$	Split the middle term. Factor each pair of terms. There is a repeated (squared) factor. There is only one root.
e Let the length be x. $x(x-2) = 15$ $x^2 - 2x - 15 = 0$ $(x-5)(x+3) = 0$ $x = 5$ or $x = -3$ The length is 5 cm and the width 3 cm.	Area = length × width Multiply out the brackets and rearrange the quadratic equation. The length cannot be negative.

You can solve quadratic equations using a polynomial application with a GDC. For example, part **d** above.



$a_1 = 4$
$a_2 = -12$
$a_3 = 9$



$a_1x^2+a_2x+a_3=0$
$x_1 = 3/2$
$x_2 = 3/2$

You will need to enter the coefficients of the quadratic expression and press SOLVE. The GDC calculates the solutions of the equation.

Solving quadratic inequalities

A quadratic inequality is similar to a quadratic equation. With two roots, you need to take care in deciding whether the solution lies between the roots or outside them.

Key point

If $a^2 < b$ then $-\sqrt{b} < a < \sqrt{b}$

If $a^2 > b$ then either $a > \sqrt{b}$ or $a < -\sqrt{b}$.

What happens when you manipulate an inequality by taking square roots of both sides?

If $x^2 < 49$ then $-7 < x < 7$ because if, for example, x was either -8 or 8 then $x^2 > 49$.

Example 25

Solve these quadratic inequalities.

a $2x^2 < 18$

b $x^2 + 2x - 8 \leq 0$ by factorization.

a $2x^2 < 18$

$x^2 < 9$

$-3 < x < 3$

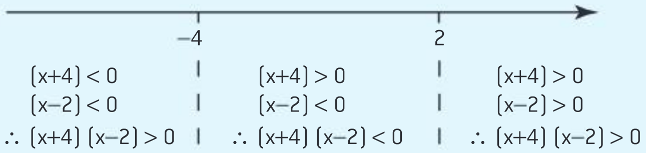
Divide both sides by 2.

Take the square roots of both sides.

b $x^2 + 2x - 8 \leq 0$

$(x+4)(x-2) \leq 0$

Factorize the quadratic expression.



$(x+4) < 0$	$(x+4) > 0$	$(x+4) > 0$
$(x-2) < 0$	$(x-2) < 0$	$(x-2) > 0$
$\therefore (x+4)(x-2) > 0$	$\therefore (x+4)(x-2) < 0$	$\therefore (x+4)(x-2) > 0$

The two roots divide the number-line into three regions. Examine the sign of the two factors in each of these regions.

Solution is $-4 \leq x \leq 2$

Exercise 2.8

In this exercise, you can verify your answers using a GDC.

1 Solve by factorizing.

a $x^2 + 5x = 0$ **b** $3x^2 - 2x = 0$ **c** $x^2 - 9 = 0$ **d** $4 - 25x^2 = 0$

e $x^2 + 7x + 10 = 0$ **f** $x^2 - 2x - 15 = 0$ **g** $x^2 + x - 6 = 0$ **h** $x^2 - 10x + 21 = 0$

i $4x^2 - 3x - 10 = 0$ **j** $6x^2 - x - 1 = 0$ **k** $9x^2 - 24x + 16 = 0$ **l** $3x^2 + 19x - 14 = 0$

2 The product of two consecutive numbers is 210. Find the numbers.

3 Solve the inequality $x^2 - 6x - 7 \geq 0$.

DP style **Analysis and Approaches HL**

4 **a** **i** Factorize $x^2 + 4x - 5$.

ii Hence solve $x^2 + 4x - 5 = 0$.

b Prove that $(x+p)^2 \equiv x^2 + 2px + p^2$.

Consider the equation $x^2 + 2px + q = 0$.

c Use your answer to **b** to write the equation in the form

$$(x+p)^2 = r$$

where r is a function of p and q.

d Hence find possible values for x in terms of p and q which solve the original equation.

e Verify that your formula will give the same answer for $x^2 + 4x - 5 = 0$ as obtained in part **a**.

2.9 Linear equations in two variables

Investigation 2.2

Aisha buys 10 packets of batteries. After opening the packets, she discovers that some packets contained two batteries and some contained three. She counts the batteries and there are 24. She wants to know how many of each type of packet she bought.

	2 batteries (x)										
3 batteries (y)	0	1	2	3	4	5	6	7	8	9	10
0											
1				9							
2											
3											
4											
5									31		
6											
7		23									
8											
9											
10											

	2 batteries (x)										
3 batteries (y)	0	1	2	3	4	5	6	7	8	9	10
0											
1		2									
2											
3											
4								11			
5											
6				9							
7											
8											
9											
10											

There are x packets with 2 batteries and y packets with 3.

The first table shows the total number of batteries with different combinations of x and y. So, for example, if $x = 3$ and $y = 1$, there is a total of 9, if $x = 8$ and $y = 5$, then the total is 31 and if $x = 1$ and $y = 7$, then the total is 23.

Complete the table, looking for combinations which result in a total of 24 batteries.

The second table shows the total number of packets with different combinations of x and y. For example, if $x = 1$ and $y = 1$, then the total is 2, if $x = 7$ and $y = 4$, then the total is 11 and if $x = 3$ and $y = 6$, then the total is 9.

Complete the table looking for combinations that result in a total of 10 packets.

Look at the combinations you have found. Which combination is in both lists? This is the number of each type that Aisha bought.

The example in investigation 2.2 consists of two equations in two variables. You could write these equations as $2x + 3y = 24$ and $x + y = 10$. These equations are called **simultaneous equations**.

The next example shows how to solve a pair of simultaneous equations by first **eliminating** one of the variables.

Key point

You can combine two equations by adding or subtracting them. If the left-hand sides are equal to the right-hand sides, then the sum (or difference) of the sides are also equal. To eliminate one of the variables, there must be the same number of x's or y's.

Example 26

a Solve $5x + 3y = 29$
$2x + 3y = 17$

b Solve $2x + 3y = 16$
$4x - 3y = 14$

c Solve $3x + 2y = 5$
$4x - 4y = 20$

a

$$\begin{aligned} 5x+3y&=29\\ 2x+3y&=17\\ \hline 3x&=12\\ x&=4 \end{aligned}$$

Since $3y$ is in both equations, subtract the second from the first.
Now you have an equation involving only x.
Solve this to find x.

$$\begin{aligned} 5\times 4+3y&=29\\ 20+3y&=29\\ 3y&=9\\ y&=3 \end{aligned}$$

Substitute for x in the first equation and find y.

b

$$\begin{aligned} 2x+3y&=16\\ 4x-3y&=14\\ \hline 6x&=30\\ x&=5 \end{aligned}$$

There are the same number of y's but they are positive in the first equation and negative in the second. To eliminate them you must *add* the equations.

$$\begin{aligned} 10+3y&=16\\ 3y&=6\\ y&=2 \end{aligned}$$

Substitute for x in the first equation and find y.

c

$$\begin{aligned} 3x+2y&=5\\ 4x-4y&=20\\ 6x+4y&=10\\ \hline 10x&=30\\ x&=3 \end{aligned}$$

Multiply the first equation by 2 so there are the same number of y's in each equation
Add the equations to eliminate y.

$$\begin{aligned} 9+2y&=5\\ 2y&=-4\\ y&=-2 \end{aligned}$$

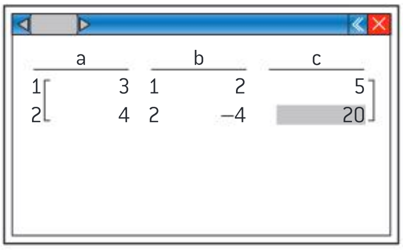
Substitute for x in the first equation and find y.

Your GDC has an application to solve systems of simultaneous linear equations. The first step is to choose 2 as the number of unknowns.

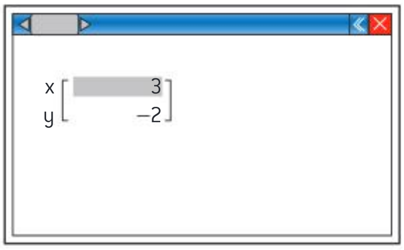
Depending on which type of GDC you are using, you will either enter the equations exactly as written or you will have to enter just the coefficients in a matrix.

To enter the coefficients as a matrix, equations need to be in the form $\begin{aligned}a_1x+b_1y=c_1\\ a_2x+b_2y=c_2\end{aligned}$ and you enter the coefficients $\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2 \end{bmatrix}$. Then press SOLVE to solve the equations.

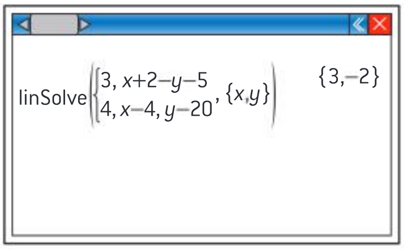
The equations in example 26 could be solved with a GDC (some calculators do this a little differently as in the screen on the right).



```
     a          b          c
1[       3  1       2          5]
2[       4  2      −4         20]
```



```
x[   3]
y[  −2]
```



$\text{linSolve}\left(\begin{cases}3, x+2-y-5\\ 4, x-4, y-20\end{cases}, \{x,y\}\right)$ $\{3,-2\}$

Exercise 2.9a

In this exercise, you can verify your answers using a GDC.

1 Solve these equations.

a $5x + 2y = 12$
$3x + 2y = 8$

b $8x + 3y = 18$
$2x + 3y = 0$

c $x + 5y = 3$
$x + 3y = 1$

d $5x - 3y = 2$
$2x - 3y = 8$

e $3x + 4y = 19$
$2x - 4y = 6$

f $5x - 2y = 1$
$4x + 2y = 26$

g $x + 3y = 10$
$2x + 5y = 18$

h $3x + 2y = 3$
$4x + 4y = 12$

i $5x - 3y = 44$
$2x + y = 11$

j $8x - 5y = 3$
$2x + 5y = 7$

k $x - y = 1$
$x + y = 5$

l $6x + 4y = 26$
$2x + 2y = 8$

Consider the equations $2x + 3y = 8$ and $5x + 2y = 9$. There is no simple way to multiply one of the equations by a number to make the coefficients equal so that you can eliminate a variable. In these circumstances you will need to manipulate both equations.

Example 27

Solve $2x + 3y = 8$
$5x + 2y = 9$

$2x + 3y = 8$ $5x + 2y = 9$ $4x + 6y = 16$ $\underline{15x + 6y = 27}$	Multiply the first equation by 2 and the second equation by 3.
$11x \quad = 11$ $x = 1$	Subtract the first equation from the second so that the coefficients are positive.
$2 + 3y = 8$ $3y = 6$ $y = 2$	Substitute for x in the first equation.

If the equations are not arranged in the standard way you can manipulate them so that they are. Alternatively, you can solve them by the method of **substitution**. In the next example, you will see both methods.

Example 28

Solve $2x + 3y = 1$
$y = 4x - 23$

Method 1: Elimination	
$2x + 3y = 1$ $y = 4x - 23$	
$-4x + y = -23$ $-12x + 3y = -69$	Subtract $4x$ from both sides of the equation. Multiply the new equation by 3.
$\underline{2x + 3y = 1}$ $-14x \quad = -70$ $x = 5$	Subtract.
$y = 20 - 23$ $y = -3$	Substitute for x in the second equation.

Method 2: Substitution

Substitute for y in the first equation and solve for x.

$$2x + 3y = 1$$
$$y = 4x - 23$$
$$2x + 3(4x - 23) = 1$$
$$2x + 12x - 69 = 1$$
$$14x = 70$$
$$x = 5$$

Substitute for x in the second equation.

$$y = 20 - 23$$
$$y = -3$$

Exercise 2.9b

In this exercise, you can check your answers using a GDC.

1 Solve these equations.

a $4x + 2y = 18$, $5x + 3y = 25$

b $7x - 4y = 22$, $6x + 3y = 6$

c $x = 5y - 3$, $2x + 3y = 20$

d $2x + 7y = 2$, $y = 2x + 14$

Higher Level

DP style **Analysis and Approaches HL**

2 **a** Explain why there is no solution to the system of equations:

$$x + 2y = 7$$
$$2x + 4y = 10$$

Consider the system of equations given by:

$$x + ay = 7$$
$$2x + 3y = b$$

b Solve the equations to find values of x and y in terms of a and b.

c State the value of a for which there is no unique solution for x and y. Explain with reference to the original equation why this is the case.

d For this value of a there is one value of b for which there is an infinite number of solutions.

- **i** State this value of b
- **ii** For these values of a and b, give two possible solutions to the system of equations.

DP style **Applications and Interpretation HL**

3 A population of foxes and a population of rabbits live on an island. The number of foxes on the island is f and the number of rabbits is r.

Space and resources on the island are limited so a model is proposed such that

$$f + 3r \leq 240$$

a For this model find the maximum number of rabbits the island could support and the number of foxes that would allow this.

Because the foxes sometimes eat rabbits another constraint on the population is that the number of rabbits must always be at least 5 times the number of foxes.

b Write this constraint as an inequality

c Write both inequalities as equations and find the values of f and r that satisfy these equations simultaneously.

d Explain why this amount of foxes and rabbits maximizes the total number of animals on the island.

Chapter summary

- ☐ You can use letters in place of numbers in expressions and formulae and evaluate the expression by substituting these letters with the values that they represent.
- ☐ The **distributive law** states that $a(b+c)=ab+ac$.
- ☐ If $x=y$ then $x+b=y+b$ and $ax=ay$.
- ☐ **Algebraic fractions** have a numerator and a denominator like a numerical fraction, but the denominator or the numerator, or both, is an algebraic term.
- ☐ An equation is a mathematical expression stating that two or more quantities are the same as one another.
- ☐ You can solve inequalities by algebraic manipulation:
  - ☐ adding or subtracting the same number to both sides does not change the inequality
  - ☐ multiplying or dividing both sides by the same *positive* number does not change the inequality
  - ☐ multiplying or dividing both sides by the same *negative* number reverses the direction of the inequality.
- ☐ A quadratic equation has terms in x^2, x and a constant term. You can arrange it in the form $ax^2+bx+c=0$.
- ☐ If $a^2<b$ then $-\sqrt{b}<a<\sqrt{b}$, and if $a^2>b$ then either $a>\sqrt{b}$ or $a<-\sqrt{b}$.

Chapter 2 test

1 If $r=-3$, $s=4$ and $t=2$, find the value of:

a $s+3t$ **b** $r+s+t$

c $t(s^2+r)$ **d** $\dfrac{\sqrt{r+s}}{t}$

2 Expand:

a $4(a-3)$ **b** $x(2x-3)$

Expand and simplify:

c $3(4-2x)+2(5x-2)$

d $\dfrac{3}{x}(x+3x^2)+x(1+4x)$

3 Factorize:

a $6x+4$ **b** $3x^2-9x$

c $8x^2y^3+12xy^4z$ **d** $\dfrac{2x^3}{3}-\dfrac{3x^2}{4}$

4 The perimeter, P, of a rectangle is given by the formula $P=2(a+b)$ where a is the width and b is the length. Rearrange this formula to make a the subject.

5 The conservation of energy formula states that $E=\frac{1}{2}mv^2+mgh$. Factorize the right-hand side of the formula and change the subject to m.

6 Draw a diagram to show the inequality $-3\le x<2$.

7 Solve the equations:

a $3(x-2)=15$ **b** $\frac{1}{2}(x-4)+\frac{1}{3}(x+3)=4$

8 Solve the inequality $5x-3\le 3(x+2)$.

9 Factorize $x^2-4x-21$.

10 Solve $4x+3y=13$
$2x-3y=11$

DP style **Applications and Interpretation SL**

11 The daily cost (\$C) to a manufacturer of producing n chairs is given by the equation

$C = 500 + 25n$.

a Find the cost of producing 10 chairs.

b **i** Rearrange the expression to make n the subject.

ii Hence find the number of chairs can be made for a daily cost of \$900.

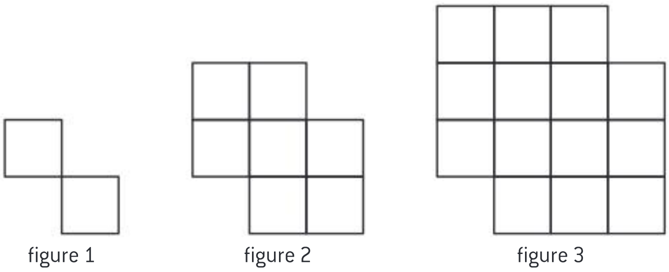
The chairs can be sold for \$50 each.

c Write down an expression for the money received when n chairs are sold.

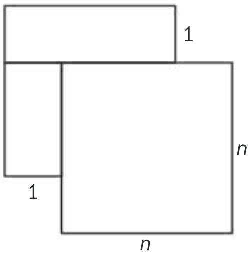
d Find the number of chairs which would need to be sold each day if the money made was to equal the daily costs.

DP style **Analysis and Approaches HL**

12 A teacher asks two of her students, Alice and Bernard, to find a formula for the number of small squares (N) in the n^{th} figure for the pattern below.



Alice decides to count the small squares by first considering the large lower square (as shown) and then adding on the number of small squares in two rectangles.



a Use the diagram above to explain why, for the n^{th} figure, Alice's formula will be $N = n^2 + 2n - 1$.

Bernard has a different approach. He decides to count the squares by thinking of each figure as a large square with two small squares removed from the top right and bottom left corner.

b Write down an expression for the number of small squares in the n^{th} figure using Bernard's method.

c Prove Bernard's expression is equivalent to Alice's.

13 In simple harmonic motion, $v = \pm\omega\sqrt{A^2 - x^2}$. Change the subject to make x the subject.

14 The formula $T = 2\pi\sqrt{\frac{l}{g}}$ gives the period, T of a simple pendulum. Rearrange the formula to make l the subject.

Modelling and investigation

DP ready **Approaches to learning**

Critical thinking: analyzing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

Self-management: Managing time and tasks effectively

A Soccer league consists of a number of teams that play each other. During the soccer season, each team plays each of the other teams in the league twice, once at home and once away. The league table records the matches during the season with points awarded to teams for wins and draws. The team with the most points at the end of the season is the league champion.

Alessandra is a keen fan of La Liga, the Spanish soccer league. She knows that there are 20 teams in the league and she wants to know how many matches each team played and how many there are in total during a season.

How many matches did each team play during a season?

How many matches would a team play if there were 18 or 22 teams in the league?

Write a formula to calculate the number of matches, m, played by a single team in a league with n teams.

Alessandra now decides to work out the total number of matches played in the league. Because the numbers are big, Alessandra begins by looking at much simpler problems. She investigates leagues where there are 2, 3, and 4 teams to see if she can find any patterns.

Make lists of all the matches played in leagues with these number of teams and then copy and complete this table:

Number of teams	Total number of matches
2	
3	
4	

Be organised when you make a list. Avoid making lists like this:

A vs B, B vs C, C vs B, B vs A, C vs A, etc.

You will find it difficult to avoid missing matches and even more difficult to ensure that you have included all of them.

Something like this is better:

A vs B, A vs C,
B vs A, B vs C,
C vs A, C vs B.

or a table like this

	A	B	C
A		AB	AC
B	BA		BC
C	CA	CB	

Use the pattern you have found to **predict** the number of matches played if there were five teams. Verify your prediction.

Can you show the pattern using algebra?

You will need a pattern that links the number of teams to the number of matches.

Command term

Predict means you should give an expected result.

Can you write an expression for the number of matches when there are n teams? You should justify your formula.

The first season of La Liga was in 1929 when there were only 10 clubs. How many matches were played in that season?

How many matches did Alessandra find that there were in the current season with 20 clubs?

This table shows the total number of matches played in the early years of La Liga.

Year	Matches
1934 – 1941	132
1941 – 1950	182
1950 – 1971	240
1971 – 1980	306

Let n be the number of teams in the league. Rearrange your formula above to make n the subject, and hence find the number of teams in the league for each of the years given in the table.

NUMBER AND ALGEBRA

3

Units and ratio

Learning outcomes

In this chapter you will learn about:

- SI (Système International) units for mass, time and length
- Conversion between units of area and volume
- Applications of ratio, percentage and proportion
- Commonly accepted world currencies

Key terms

- Metric system
- Capacity
- Ratio
- Unitary ratio
- Scale
- Percentage
- Directly proportional
- Exchange rate

3.1 Système International

DP ready **International-mindedness**

After the French Revolution in 1789, the French National Assembly introduced the **metric system** with units of mass and length. Earlier systems were chaotic and confusing, but the metric system had larger and smaller units based on multiples of 10.

The Système Internationale d'Unités was an extension to the metric system and was adopted by the International Bureau of Weights and Measures in 1960 to satisfy the needs of the scientific and technological community.

The unit of mass is the kilogram (kg) and the unit of length is the metre (m). Prefixes such as kilo-, mega-, centi-, milli-, and micro- are used to change the base units into larger and smaller units. The term kilogram is itself a combination of the prefix kilo- meaning 1000 and gram, so 1 kg = 1000 g. Each prefix has a symbol assigned to it.

This table lists some of the more commonly used prefixes and their symbols. To convert a measurement to another unit, you need to multiply or divide by a power of 10, as shown in the table.

	×1000		×1000		×1000		×1000		×100		×10		×1000		×1000	
tera-		giga-		mega-		kilo-		base unit		centi-		milli-		micro-		nano-
T		G		M		k				c		m		μ		n
	÷1000		÷1000		÷1000		÷1000		÷100		÷10		÷1000		÷1000	

Altogether there are seven SI base units, but the ones that you are likely to come across are metre (distance), kilogram (mass), and second (time), together with their multiples according to the prefixes and units based on a combination of these.

Example 1

Convert:

a 3 kg to g **b** 4 mm to cm **c** 5 km to cm **d** 7 µg to g

a 3 kg = 3 × 1000 = 3000 g	kilo to base unit (× 1000)
b 4 mm = 4 ÷ 10 = 0.4 cm	milli to centi (÷ 10)
c 5 km = 5 × 1000 = 5000 m	convert kilo to base unit (× 1000)
= 5000 × 100 = 500 000 cm	and base unit to centi (× 100)
= 5×10^5 cm	
d 7 µg = 7 ÷ 1 000 = 0.007 mg	convert micro to milli (÷ 1000)
= 0.007 ÷ 10 = 0.0007 cg	milli to centi (÷ 10)
= 0.0007 ÷ 100 = 0.000 007 g	and centi to base unit (÷ 100)
= 7×10^{-6} g	

There are many other units that you can derive from the base units. For example, since speed = $\frac{\text{distance}}{\text{time}}$, then the units of speed are metres (the unit of distance) per second (the unit of time), sometimes written m/s or more usually m s^{-1}.

Exercise 3.1

1 Convert:

a 1.5 cm to m **b** 0.034 kg to g **c** 1500 mm to km **d** 234 mg to g

e 0.0001 mm to nm **f** 24.5 g to kg **g** 0.0012 km to cm **h** 275 mm to cm

i 24 mg to µg **j** 3.12×10^6 µg to kg **k** 2.3×10^3 cm to km **l** 4.3×10^{-4} kg to mg.

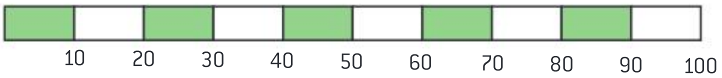
2 A tonne is 1000 kg. Find how many 30 kg boxes can be loaded on a truck that will carry a maximum of 21 tonnes.

3 A camera flash illuminates for 1000 microseconds. Calculate what fraction of a second this is.

4 Your fingernail will grow approximately 1 nanometer in a second. How many millimeters will it grow in a week?

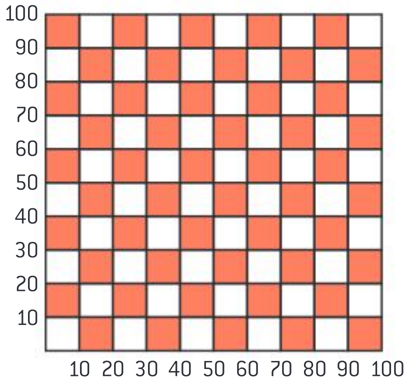
5 Computer processor speed is measured in Hertz (Hz). In 1993 a processor had a speed of 60 MHz. Processors in 2018 were capable of 3 GHz. Calculate how many times faster this is.

3.2 Units of area and volume

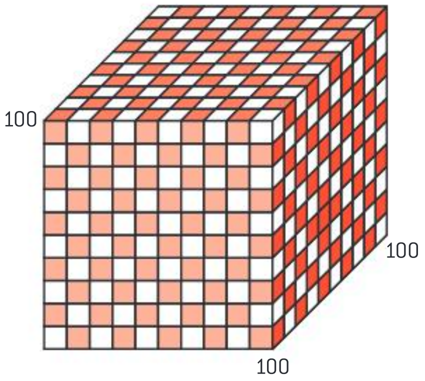


There are 100 centimeters in 1 m.

There are 100 × 100 = 10 000 1 cm^2 squares in a 1 m^2 square.



There are 100 × 100 × 100 = 1 000 000 1 cm^3 cubes in a 1 m^3 cube.



Summarizing these results:
1 m = 100 cm,
1 m^2 = 10 000 cm^2 and
1 m^3 = 1 000 000 cm^3.

Doing the same thing with mm:
1 m = 1000 mm,
1 m^2 = 1000 × 1000 = 1 000 000 mm^2 and
1 m^3 = 1000 × 1000 × 1000 = 1 000 000 000 mm^3.

Internal link

Yes! One million cubic centimetres make a cubic metre. You will learn how to find the volume of a cube in chapter 7.

DP ready Theory of knowledge

This can give you an idea of just how big a million is. Think of a space that is 1 m^3. For example, the space under your school desk might be approximately 1 m^3. Now think of something that is approximately 1 cm^3. An example of this could be a sugar cube. So, if you had to pack the space under your school desk with sugar cubes, there would be 1 million cubes. (If you were to consume the recommended intake of sugar every day, it would take over 350 years to eat all this sugar.)

Microbeads, which are tiny plastic beads used in some beauty products, are about 1 mm^3. If you filled the space under your desk with these there would be one billion of them.

Key point

$$1\text{ m} = 100\text{ cm},\ 1\text{ m}^2 = 100^2\text{ cm}^2 = 10\,000\text{ cm}^2,$$
$$1\text{ m}^3 = 100^3\text{ cm}^3 = 1\,000\,000\text{ cm}^3.$$
$$1\text{ cm} = \frac{1}{100}\text{ m},\ 1\text{ cm}^2 = \frac{1}{100^2}\text{ m}^2 = \frac{1}{10\,000}\text{ m}^2,$$
$$1\text{ cm}^3 = \frac{1}{100^3}\text{ m}^3 = \frac{1}{1\,000\,000}\text{ m}^3.$$

Volume is the space that something occupies and **capacity** is the amount that something can contain. These definitions are similar and are clearly related to each other and you measure them in the same way. The unit of capacity is the litre (l) where 1 l = 1000 cm^3.

Key point

$$1\text{ l} = 10^3\text{ cm}^3,$$
$$1\text{ cm}^3 = \frac{1}{1000}\text{ litres} = 1\text{ ml}.$$

Example 2

a The volume of a cuboid is 720 000 cm^3. Convert this volume to m^3.

b The volume of a cuboid is 0.92 m^3. Calculate its volume in litres.

a $720\,000\text{ cm}^3 = 720\,000 \div 1\,000\,000\text{ m}^3$ $= 0.72\text{ m}^3$	Using 1 m^3 = 1 000 000 cm^3.
b $0.92\text{ m}^3 = 0.92 \times 1\,000\,000\text{ cm}^3$ $= 920\,000\text{ cm}^3$ $= 920\,000\text{ ml}$ $= 920\text{ litres}$	Using 1 m^3 = 1 000 000 cm^3. 1 cm^3 = 1 ml 1000 ml = 1 litre

Example 3

a A carton contains 33 centilitres (cl) of drink. Given that 100 cl = 1 l, find the volume of juice in the box in cm^3.

b How many of these cartons could you pack into a box with a volume of 0.5 m^3? Give your answer to 2 s.f.

a $33 \text{ cl} = 33 \div 100 \text{ l}$ $= 0.33 \text{ l}$	Using $100 \text{ cl} = 1 \text{ l}$
$= 0.33 \times 1000 \text{ cm}^3$ $= 330 \text{ cm}^3$	Using $1000 \text{ cm}^3 = 1 \text{ l}$
b $0.5 \text{ m}^3 = 0.5 \times 1\,000\,000 \text{ cm}^3$ $= 500\,000 \text{ cm}^3$ $500\,000 \div 330 = 1500$ There will be 1500 cartons.	Assuming that you can pack the cartons in the box with no wasted space.

Exercise 3.2

1 Convert:

a 0.0279 m^2 to cm^2 **b** 0.5 m^2 to mm^2 **c** 23 cm^2 to mm^2 **d** 343 cm^2 to m^2

e $40\,000 \text{ mm}^2$ to m^2 **f** 1.5 m^2 to cm^2 **g** 0.025 mm^2 to cm^2 **h** 12.2 cm^2 to mm^2

2 Convert:

a 0.00125 m^3 to cm^3 **b** 456 cm^3 to m^3 **c** 1.21 m^3 to cm^3 **d** 0.0076 cm^3 to mm^3

e 0.12 cm^3 to mm^3 **f** 0.32 m^3 to mm^3 **g** 340 mm^3 to cm^3 **h** 0.0003 m^3 to mm^3

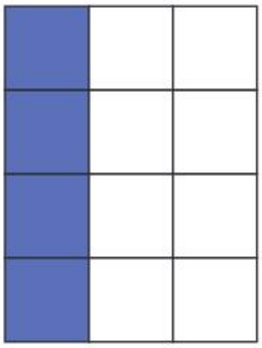
3 A dose of medicine is one 5 ml spoonful. A bottle contains 175 cm^3 of the medicine. Calculate how many doses can be made from the bottle.

4 Car engines are sometimes rated in cc, where cc means cm^3. Determine the rating of a 1300 cc engine in litres.

5 1 m^3 of sand is packed in bags. Each bag has a capacity of 5 l. Find how many bags are filled.

3.3 Ratio

Understanding ratios



In the diagram there are 12 squares. 4 squares are grey and 8 squares are white.

The **ratio** of grey squares to white squares is 4 : 8.

Fractions and ratios are related. In the diagram above, $\frac{4}{12}$ or $\frac{1}{3}$ of the squares are grey and $\frac{8}{12}$ or $\frac{2}{3}$ of the squares are white.

Example 4

The ratio of red squares to white squares is 3 : 5. What fraction of the squares are red?

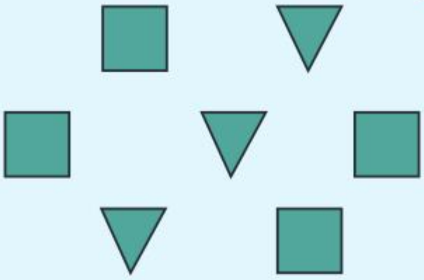
 $3 + 5 = 8$	For every 3 red squares, there are 5 white squares. Add the number of white and red to give the total number of squares.
So $\frac{3}{8}$ of the squares are red.	Express the number of red squares as a fraction of the total squares.

Example 5

a What is the ratio of squares to triangles?

b What is the ratio of triangles to squares?

c What is the ratio of squares to shapes?



a 4 : 3	There are 4 squares and 3 triangles
b 3 : 4	
c 4 : 7	There are 4 + 3 = 7 shapes altogether.

Key point

The ratio of two numbers p and q is $p : q$. A ratio shows the relative sizes of two quantities.

Like fractions, you normally express ratios in their simplest terms.

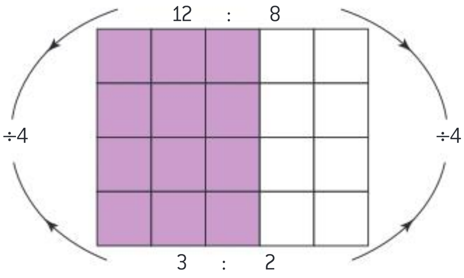
If you multiply or divide both numbers in a ratio by the same constant, the value of the ratio does not change. For example, the ratio of 12 : 8 is equivalent to 3 : 2.

Internal link

You learned about fractions in section 1.8

Note

Would a ratio remain the same if you added a constant to both sides, or if you subtract a constant from both sides?



You should simplify ratios in the same way as you simplify fractions.

Normally ratios are written using integers and in their simplest terms.

Ratios in which one of the figures is one are called **unitary ratios**. These can be written using figures that are not integers.

For example,

$$\begin{tikzcd} 4 \arrow[d, bend right, "\div 4"'] & : & 5 \arrow[d, bend left, "\div 4"] \\ 1 & : & 1.25 \end{tikzcd} \qquad \begin{tikzcd} 3 \arrow[d, bend right, "\div 2"'] & : & 2 \arrow[d, bend left, "\div 2"] \\ 1.5 & : & 1 \end{tikzcd}$$

You write the ratio 4 : 5 as a unitary ratio of 1 : 1.25 or 0.8 : 1.

You write 3 : 2 as the unitary ratio 1.5 : 1 or 1 : 0.667.

Example 6

a To make a mortar you mix cement to sand in the ratio 1 : 3. (That is 1 part cement to 3 parts sand). If you use 5 buckets of cement:

i calculate how much sand would you use

ii determine how much mortar you will make.

Note

If you are given a ratio of two quantities in a real-life situation, the first quantity corresponds to the first number in the ratio, and the second quantity to the second number. Here, you mix sand to cement in the ratio 1 : 3. Sand corresponds to 1, and cement corresponds to 3.

b **i** Three fifths of the Physics class are boys. Find the ratio of boys to girls in the class.

ii Write down the ratio of boys to girls as a unitary ratio in the form 1 : n.

a **i** 1 : 3 is the same as 5 : 15 You will need 15 buckets of sand.	$\begin{tikzcd} 1 \arrow[d, bend right, "\times 5"'] & : & 3 \arrow[d, bend left, "\times 5"] \\ 5 & : & 15 \end{tikzcd}$
ii The total amount of mortar is 5 + 15 = 20 buckets.	The fractions add to make 1.
b **i** If $\frac{3}{5}$ of the class are boys then $1 - \frac{3}{5} = \frac{2}{5}$ are girls. The ratio of boys to girls is $\frac{3}{5} : \frac{2}{5} = 3 : 2$	Multiply by 5.
ii $3 : 2 = 1 : \frac{2}{3}$	Divide by 3.

Note

We wrote the unitary fraction in exact form $1 : \frac{2}{3}$. You could have written 1 : 0.666…, but this is less accurate because of the recurring decimal.

Suppose you wanted to make 20 buckets of mortar with cement to sand in the ratio 1 : 3. Working backwards, you can find out how many of each to mix.

For a ratio of 1 : 3, you will need to split the 20 buckets into groups of 1 + 3 = 4 buckets

In each group, 1 bucket is cement and 3 buckets are sand.

So in 5 groups there are 5 × 1 buckets of cement and 5 × 3 = 15 buckets of sand.

For a different mix, the ratio of cement to sand is 1 : 4. To make the same quantity, split 20 into groups of 1 + 4 = 5 buckets.

In each group, 1 bucket is cement and 4 buckets are sand.

Now there are 4 × 1 buckets of cement and 4 × 4 = 16 of sand.

Exercise 3.3a

1 Express these ratios in the form 1 : *n*.

a 2 : 6 **b** 4 : 6 **c** 6 : 2 **d** 8 : 5

2 Express these ratios in the form *n* : 1.

a 6 : 3 **b** 9 : 5 **c** 1 : 4 **d** 4 : 10

3 A box contains red and black balls. $\frac{5}{8}$ of the balls are red. What is the ratio of red balls to black balls? Write this as a unitary ratio in the form 1 : *n*.

4 Write these ratios in their simplest terms.

a 8 : 12 **b** 15 : 10 **c** 2 cm : 5 mm **d** 8 kg : 500 g

e 45 min : 2 h **f** 330 ml : 1.2 l **g** 1.5 m : 450 cm **h** 20 sec : 0.5 min

5 Dean makes up a juice drink using concentrate to water in the ratio 2 : 7. His sister makes her drink using 3 parts concentrate to 9 parts water. Whose drink is the strongest?

6 The ratio of female to male workers in the labour force was 16 : 18 in France while in Chile the ratio was 20 : 28. Which of the two countries has the greater proportion of female workers?

7 In a mortar mix, the ratio of sand to cement is 3 : 1. If I have 4 tonnes of sand, how much cement do I need?

8 Brass is an alloy made from a mixture of copper and zinc in the ratio 13 : 7. If I use 35 g of zinc, how much copper will I need? What weight of brass will this produce?

Dividing into a given ratio

You can also divide a quantity into a ratio.

Example 7

a Two brothers own a company and their shares are in the ratio 4 : 7. Their annual profit is \$132 000 and they want to split it in the ratio of their shares. Calculate how much of the profit each brother gets.

b A box of chocolates contains plain chocolates and milk chocolates in the ratio 3 : 5. If there are 32 chocolates in the box, how many plain and how many milk chocolates are there?

c Students in a school can choose physics, chemistry or biology for their group 4 subject. They choose these in the ratio 3 : 2 : 5. If there are 60 students in the IB Diploma class, how many choose each subject?

a	$132\,000 \div (4+7) = 12\,000$ $12\,000 \times 4 = \$48\,000$ $12\,000 \times 7 = \$84\,000$	Split the shares into 4 + 7 parts. One brother has 4 parts. The other has 7 parts.
b	$32 \div (3+5) = 4$ $4 \times 3 = 12$ $4 \times 5 = 20$ There are 12 plain and 20 milk chocolates	Split the shares into 3 + 5 parts. There are 3 parts plain and 5 parts milk.
c	$60 \div (3+2+5) = 6$ $6 \times 3 = 18$ $6 \times 2 = 12$ $6 \times 5 = 30$ 18 take physics, 12 take chemistry and 30 take biology	Split the shares into 3 + 2 + 5 parts. There are 3 parts physics, 2 parts chemistry and 5 parts biology.

Ratios are used for **scales** such as map scales or the scales used in models. Often these are expressed as unitary ratios in the form 1 : n. This means that 1 cm in the map or the model represents n cm in real life.

Example 8

A map has a scale of 1 : 50 000. Varun measures the distance he walks to school on the map. If the distance on the map is 2.5 cm, determine how far Varun walks to school (in km).

2.5 cm = 50 000 × 2.5 = 125 000 cm	Multiply the distance on the map by the map scale to get the real distance.
125 000 ÷ 100 000 = 1.25 km	To convert cm to km ÷ 100 and ÷ 1000

Example 9

Aurelia has a 1 : 72 scale model of a plane. If the wingspan of the real plane is 10.8 m, what is the wingspan, in cm, of the model plane.

10.8 × 100 = 1080 cm	To convert m to cm × 100.
1080 ÷ 72 = 15 cm	Divide the real wingspan by the model to get the model wingspan.

Exercise 3.3b

1 \$42 is divided in the ratio 2 : 5. What is the difference between the largest and smallest shares?

2 What is the ratio of 2 cm : 5 mm in its simplest terms? (You must make sure that both quantities are in the same units).

3 An area of 80 m^2 is divided between grass and paving in the ratio 11 : 5. What are the areas of grass and paving?

4 Jean, Michel and Boris divide some sweets in the ratio of their ages. Jean is 12 years old, Michel is 8 and Boris is 7. If they have 81 sweets, how many does each get?

5 18 carat rose gold is an alloy that is made using gold, copper and silver in the ratio 300 : 89 : 11. If a ring weighs 6 g, how much of each metal is there in the ring?

6 Hans makes a model of the tallest building in London, the Shard, to a scale of 1 : 1024. If his model is 303 mm tall, how tall, in m, is the Shard?

7 The distance between Brooklyn and Manhattan is 15 km. How far is this, in cm, on a map which is to the scale 1 : 25000?

3.4 Percentages

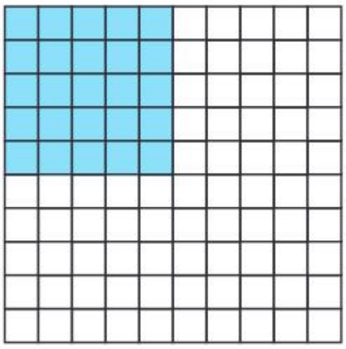
Percentage of a quantity

Key point

A **percentage** means *parts per hundred*.

For example, 25% is 25 parts per hundred.

$$25\% = \frac{25}{100} \qquad \begin{tikzcd} \frac{25}{100} \arrow[r, bend left, "\div 25"] \arrow[r, bend right, "\div 25"'] & \frac{1}{4} \end{tikzcd} \qquad 25\% = 0.25$$



Key point

A percentage can be expressed as a fraction or as a decimal.

Divide by 100 to change from a percentage to a fraction or a decimal.

Fractions and decimals can be expressed as percentages.

Multiply by 100 to change to a percentage.

$$\frac{4}{5} = \left(\frac{4}{{}_1\cancel{5}} \times \cancel{100}^{20}\right)\% = \left(\frac{4 \times 20}{1}\right)\% = 80\%$$

$$0.35 = (0.35 \times 100)\% = 35\%$$

Internal link

You learned how to simplify fractions in section 1.8.

You will often need to write one quantity as a percentage of another. For example, if you get 45 marks out of a total of 75 marks in a test, what is 45 as a percentage of 75?

To find the percentage, first write the result as a fraction and then convert to a percentage.

45 marks out of 75 is $\frac{45}{75}$.

As a percentage, this is $\frac{{}^{15}\cancel{45}}{{}_{1}\cancel{{}^{3}}\cancel{75}} \times \cancel{100}^{4} = \frac{15 \times 4}{1} = 60\%$.

Example 10

a Write 35% as a fraction.

b 2400 out of a group of 6000 students said that they preferred MAI to MAA. Write this as a percentage.

c Write 5% as a decimal.

a $\frac{\cancel{35}^{7}}{\cancel{100}_{20}} = \frac{7}{20}$	Divide by 100.
b $\frac{2400}{6000} \times 100 = 40\%$	
c $5\% = \frac{5}{100} = 0.05$	First write 5% as a fraction, and then convert this fraction to a decimal.

You will also need to find percentages of a quantity. For example, you can find 20% of \$150 as shown.

Write the percentage as a fraction or decimal and then multiply by the amount.

$20\% = 0.2$ then $150 \times 0.2 = \$30$

Example 11

Find 12% of €25.

$12\% = 0.12$	First write the percentage as a decimal number.
$0.12 \times 25 = €3$	Then multiply the quantity by the decimal.

Example 12

Find 60 g as a percentage of 1.2 kg.

$1.2\text{ kg} = 1200\text{ g}$	Ensure that the quantities are in the same units.
$\frac{60}{1200} \times 100 = 5\%$	

Exercise 3.4a

1 Write these fractions as percentages.

a $\frac{17}{20}$ **b** $\frac{2}{3}$ **c** $\frac{18}{35}$

2 Write these decimals as percentages.

a 0.34 **b** 1.75 **c** 0.675

3 Write these percentages as:

i fractions (in their lowest trems) **ii** decimals.

a 35% **b** 12.5% **c** 140%

4 **a** Leila scores 35 marks out of 56 in a Spanish test. What is her mark as a percentage?

b Kwaku scores 75% in the same test. How many marks out of 56 did he get?

5 Out of the voters in Atown, 56% voted for the Rhombus Party. If there were 125 300 voters in Atown, how many voted for the Rhombus Party?

6 A gardener buys a packet containing 750 seeds. Of these seeds, 75% will successfully germinate. Of the seeds that germinate, 64% will grow to maturity. How many mature plants can the gardener expect to grow from his packet?

Another packet states that out of 1000 seeds, 70% will germinate and 48% will grow to maturity. If both packets cost the same, which is the better buy?

DP style **Applications and Interpretation HL**

7 An isolated area of farming land has a simple food chain. Airborne insects are eaten by sparrows, and sparrows are eaten by a population of hawks.

On average one sparrow will eat 90 insects in a day, and a hawk will eat three sparrows in a day.

The area is 3 km^2 and the insect population favoured by the sparrows is stable (that is, the population size neither increases or decreases) if no more than 1% of their numbers are eaten each day. Initially there are 300 insects per m^2. The sparrow population remains stable if no more than 0.3% of their population is eaten each day by hawks.

a Find the maximum number of sparrows the area can support without a decline in the insect population.

b For this number of sparrows, find the maximum number of hawks the area can support.

The farmers spray the land with insecticide that reduces the number of airborne insects to 60 per m^2.

c Find the maximum population of sparrows which is now sustainable without the insect population decreasing.

After a short time the sparrow population has reduced to the new levels but the hawk population still remains at its previous level.

d Assuming there is no time for the sparrow population to reproduce, determine how many days from this point it will be before the population of sparrows becomes extinct.

e Comment on the effect of this extinction on the populations of insects and hawks, assuming there is no further intervention by the farmers.

Percentage increase and decrease

Percentages can be used to increase or to decrease an amount.

Example 13

If global temperatures increase by 2%, find the new average temperature of a city with previous average temperature of 25 °C.

We require 100% + 2% = 102% of 25 °C.	If something is increased by 2% then the new amount is 102% of the original.
$102\% = \frac{102}{100} = 1.02$	Change to a decimal.
$1.02 \times 25 = 25.5$ °C	To find 102% you need to multiply by 1.02.

In a similar way you can find percentage decreases.

Example 14

Find the sale price of an item that is advertised as 15% off when the original price was $32.

We require 100% – 15% = 85% of $32	After a decrease of 15%, you need to find 85% of the original amount.
$85\% = \frac{85}{100} = 0.85$	Change to a decimal.
$0.85 \times \$32 = \$27.20$	To find 85% you need to multiply by 0.85.

In percentage problems, always take care with units. If you are looking for one quantity as a percentage of another, they must both be in the same units.

Example 15

a Increase $250 by 10%.

b Decrease 60 marks by 15%.

a $1.1 \times 250 = \$275$	If 100% is increased by 10%, then it will become 110%. Multiply by 1.1.
b $60 \times 0.85 = 51$ marks	If 100% is decreased by 15% then there will be 85% left. Multiply by 0.85.

Example 16

Sven puts his money into a bank account that pays him interest so that his money increases by 5% each year. If he pays in € 240, then how much will he have after **i** 1 year **ii** 2 years?

i $240 \times 1.05 = €252$	If 100% is increased by 5%, then it will become 105%. Multiply by 1.05.
ii $252 \times 1.05 = €264.60$	To find the amount after 2 years, multiply by 1.05 again.

Higher Level

20 % off
SALE PRICE
$160

A ticket on an item in a sale states that its price has been reduced to $160 after a 20% reduction. The shopkeeper decides to remove the item from the sale and restore it to its original price. He calculates 20% of $160 which is $32 and adds this to the sale price and puts a new ticket on selling the item at \$192. Is this a correct calculation?

An assistant points out that the 20% reduction was on the *original* price, not on the sale price, so the calculation must be wrong. She argues that if the price had been reduced by 20% then the original amount, x, has been multiplied by 0.8. So to find x we need to solve $0.8x = 160$.

$$\begin{aligned}0.8x &= 160\\ x &= \frac{160}{0.8}\\ x &= 200\end{aligned}$$

Internal link

To recap solving linear equations, see section 2.5a.

Example 17

A box of biscuits has a label saying it includes 20% extra free. If it now weighs 144 g, find the original weight of the packet before 20% was added.

$1.2x = 144$ $x = \frac{144}{1.2}$ $x = 120\text{ g}$	After an increase of 20%, the packet would be 120% of the original weight.

Exercise 3.4b

1 A hotel chain increases its prices by 8%. If the price of a room was \$120, what will it be after the increase?

2 The average annual inflation rate in the US for the last century has been 3.15% per year. If the price of new car is \$36 000, how much would you expect this to rise to next year if it increases by 3.15%?

3 In a sale, all prices in a shop are decreased by 15%. If a coat was priced at \$140, what will the sale price be?

4 Tina has a loyalty card that gives her 12.5% off the cost of meals in the Binomial Burger Bar. If the advertised cost of a meal is \$ 9.60, what will Tina have to pay?

5 **a** In a clearance sale, a store reduces its prices by 30%. How much are they selling an item that previously cost \$3.70?

  b In the same sale an item is being sold for \$15.40. Calculate its original price.

6 During the year, a shop increases the price of soap powder by 10% in March and a further 10% in September. What is the overall increase during the year?

Hint

Let the original price be x. Multiply x by 1.10 for the first increase. Then multiply this number by 1.10 again for the second.

3.5 Direct proportion

Suppose an apple costs 40p. Then two apples cost 80p, and three apples cost £1.20. The total price of the apples increases by the same amount each time as the number of apples increases.

We say that the number of apples is **directly proportional** to the total price of the apples.

Key point

If a quantity y is a constant multiple of another quantity x, then y is **directly proportional** to x.

Example 18

12 oranges cost €13.20. Assuming each orange costs the same amount:

a calculate the price of 36 oranges

b find how many oranges you could buy for €16.50.

a $36 \div 12 = 3$

$$\begin{tikzcd} x & y \\ \text{12 oranges} \arrow[d, bend right, "\times 3"'] & \text{€13.20} \arrow[d, bend left, "\times 3"] \\ \text{36 oranges} & \text{€39.60} \end{tikzcd}$$

The cost of 36 would be €39.60.

Find the number which you multiply 12 by to get 36.

If $12 \times 3 = 36$ then $13.20 \times 3 = 39.60$.

b $16.50 \div 13.20 = 1.25$

$$\begin{tikzcd} x & y \\ \text{12 oranges} \arrow[d, bend right, "\times 1.25"'] & \text{€13.20} \arrow[d, bend left, "\times 1.25"] \\ \text{15 oranges} & \text{€16.50} \end{tikzcd}$$

You could buy 15 oranges for €16.50.

Find the number you multiply 13.20 by to get 16.50.

If $13.20 \times 1.25 = 16.50$ then $12 \times 1.25 = 15$.

Exercise 3.5

In this exercise you should assume that the quantities are in direct proportion.

1 20 l of petrol costs £24. What is the cost of 40 l?

2 If Maria can earn \$75 for 6 hours work, how much would she earn for 18 hours work?

3 If I can walk 3 km in 45 minutes, how far could I walk in 1 hour?

4 What would be the cost of 200 g of flour if 1 kg costs \$7.

5 24 pencils cost \$4.56. What is the cost of 50?

6 A piece of apple pie in a restaurant costs €4.50. Instead of buying a piece I decide to make an apple pie at home.

The main ingredients in the recipe I will use are as follows. The recipe will be large enough to make 6 pieces.

160g sugar
5 g ground cinnamon
600 g cooking apples
400 g plain flour
200 g butter cut into cubes
1 medium egg

The costs of the ingredients at the supermarket are:

Sugar 75¢ for 1 kg
Cinnamon 80¢ for 40 g

Cooking apples €1.85 per kilogram
Plain flour €1.80 for 1 kg
Butter €1.50 for 250 g
Eggs €1.80 for 12

a Find the cost of making the apple pie.

b Express the cost of making the pie as a percentage of the cost in the restaurant.

DP style **Analysis and Approaches HL**

7 An alternative way to solve proportion problems is to write the unknown term as x and use the fact that the ratio of the two quantities is always equal.

For example if x and y are directly proportional and if $y = 12$ when $x = 5$ and we need to find the value of x when y is equal to 15, we can write the proportion, $\frac{x}{y}$, as $\frac{x}{15} = \frac{5}{12}$.

a Solve the equation given above to find the value of x.

b Find the value of y when x is equal to 9.

x and y are directly proportional. It is given that when $x = a$ $y = 6$ and when $x = 2$ $y = a + 1$.

c Find the value of a.

DP style **Applications and Interpretation HL**

8 In order to work out the number of fish in a lake, a biologist catches a sample of 60 fish and tags them.

A week later he returns to the lake and catches a sample of 80 fish and finds 12 of these have been tagged.

Use proportions to find an estimate for the number of fish in the lake, stating any assumptions you are making.

Key point

The **exchange rate** between a pair of currencies is how much one currency will buy of another. Exchange rates change by the second.

3.6 Currency

'Currency trading' is the buying and selling of currencies in the foreign exchange market.

The table shows how much of each currency you would need to buy 1 USD. The rates will vary over time.

Currencies used by the top 20 economies in the world on 13 Feb 2019

Country	Currency	Symbol	Code	Exchange rate (1 USD)
Australia	Dollar	$	AUD	1.41
Brazil	Real	R$	BRL	3.72
Canada	Dollar	$	CAD	1.32
China	Renminbi (Yuan)	¥	CNY	6.76
Eurozone	Euro	€	EUR	0.88
India	Rupee	₹	INR	70.81
Indonesia	Rupiah	Rp	IDR	14 056
Japan	Yen	¥	JPY	110.71
Mexico	Peso	$	MXN	19.34
Russia	Rouble	₽	RUB	65.73
Saudi Arabia	Riyal	ريال	SAR	3.75
South Korea	Won	₩	KRW	1 123.40
Switzerland	Franc	CHF	CHF	1.01
Turkey	Lira	₺	TRY	5.25
United Kingdom	Pound	£	GBP	0.77
United States	Dollar	$	USD	1.00

Most currencies can be subdivided into 100 parts (cents, etc.). When writing a currency amount you should give your answer as an exact amount or to 2 d.p.

Do not round amounts of money to 3 s.f. If your average monthly salary was \$2 773.50 and your employer rounded this to 3 s.f. you would only receive \$2 770, and might not be very happy!

Example 19

a Use the table to convert 50 USD to these currencies:

i BRL **ii** KRW

b Use the table above to convert the following amounts to US dollars (USD):

i 27.40 GBP **ii** 100 000 IDR

a i

$$\begin{tikzcd} 1\text{ USD} \arrow[d, bend right, "\times 50"'] & 3.72\text{ BRL} \arrow[d, bend left, "\times 50"] \\ 50\text{ USD} & 186\text{ BRL} \end{tikzcd}$$

When looking at your answer, ask yourself whether you would have more dollars or more of the currency. This is a good way to check your answer.

$3.72 \times 50 = 186$ BRL

ii

$$\begin{tikzcd} 1\text{ USD} \arrow[d, bend right, "\times 50"'] & 1123.40\text{ KRW} \arrow[d, bend left, "\times 50"] \\ 50\text{ USD} & 56170\text{ KRW} \end{tikzcd}$$

$50 \times 1123.40 = 56170$ KRW

b i

$$\begin{tikzcd} 0.77\text{ GBP} \arrow[d, bend right, "\times 35.58"'] & 1\text{ USD} \arrow[d, bend left, "\times 35.58"] \\ 27.40\text{ GBP} & 35.58\text{ USD} \end{tikzcd}$$

Find the multiplier

$27.40 \div 0.77 = 35.58$

ii

$$\begin{tikzcd} 14\,056\text{ IDR} \arrow[d, bend right, "\times 7.11"'] & 1\text{ USD} \arrow[d, bend left, "\times 7.11"] \\ 100\,000\text{ IDR} & 7.11\text{ USD} \end{tikzcd}$$

Find the multiplier

$100\,000 \div 14\,056 = 7.11$

Investigation 3.1

In order to compare the cost of living in various countries, Michel collects information about the average cost of a 1.5 l bottle of water. Here are the results for six countries.

Country	Cost
China	4.34 CNY
Indonesia	5 058 IDR
Japan	155.80 JPY
Russia	57.84 RUB
Switzerland	1.20 CHF
United States	1.75 USD

How can you compare the cost in these countries?

Assuming the cost of a bottle of water is representative of the cost of living in general, use the data to rank these countries in order of the cost of living.

To find whether the cost of water is representative of the cost of living, Michel looks at another factor: the cost of a pair of jeans in each country.

Country	Cost
China	212.32 CNY
Indonesia	439 795 IDR
Japan	6576.70 JPY
Russia	4504.70 RUB
Switzerland	110.98 CHF
United States	44.59 USD

Rank the countries in the same way according to cost of jeans. Do you get the same ranking? How would you account for any differences?

Exercise 3.6

1 Using the table at the beginning of section 3.4, convert:

a 100 SAR to USD **b** 1000 MXN to USD **c** 43 USD to JPY **d** 0.30 USD to INR

e 72 EUR to USD **f** 1.25 USD to TRY **g** 1850 USD to CAD **h** 6 BRL to USD.

2 Léonie is travelling from Switzerland to Turkey for a holiday. She changes 1250 CHF into Lira. The exchange rate is 1 CHF to 5.23 TRY.

a Calculate how much Léonie receives in TRY.

While on holiday she spends 5000 Lira. When she returns, Léonie's bank will change any banknotes over 50 TRY back to CHF.

b Find the maximum amount of money, in TRY, that she could change back.

The exchange rate that Léonie gets for changing the money back is 5.58 TRY to 1 CHF.

c Calculate how much money, in CHF, she receives for exchanging back the unused banknotes.

DP style **Analysis and Approaches SL**

3 Safa has just arrived in Germany from the United States. She fills her car with petrol and notices that 40 litres costs her €53.60. She remembers that just before she left the United States she filled her 13 gallon tank for $60.00.

Supposing that fuel prices are the same in the USA as they are in Germany and that 1 gallon is approximately 3.79 litres, find the exchange rate at the time. Give your answer in the form $1 : €$x$.

Chapter summary

☐ 1 m = 100 cm, 1 m^2 = 100^2 cm^2 = 10 000 cm^2, 1 m^3 = 100^3 cm^3 = 1 000 000 cm^3.

☐ 1 cm = $\frac{1}{100}$ m, 1 cm^2 = $\frac{1}{100^2}$ m^2 = $\frac{1}{10000}$ m^2, 1 cm^3 = $\frac{1}{100^3}$ m^3 $\frac{1}{1000000}$ m^3.

☐ 1 l = 10^3 cm^3, 1 cm^3 = $\frac{1}{1000}$ litres = 1 ml.

☐ The ratio of two numbers p and q is $p : q$. A ratio is used to show the relative sizes of two quantities.

☐ A percentage can be expressed as a fraction or as a decimal. Divide by 100 to change from a percentage to a fraction or a decimal.

☐ If a quantity y is a constant multiple of another quantity x, then y is **directly proportional** to x.

☐ The **exchange rate** between a pair of currencies is how much one currency will buy of another.

Chapter 3 test

1 Convert:

a 2.34 km to m **b** 3 250 mm to m **c** 250 nm to mm **d** 125 g to kg.

2 1 tonne (t) is 1000 kg. A container is loaded with 48 000 cans of tomatoes which weigh 454 g. Calculate the weight of these cans in tonnes.

3 A sheet of metal has an area of 1.94 m^2. It is cut into 5000 identical pieces with no wastage.

a Find the area of each piece of metal after they have been cut.

b If the whole sheet of metal weighs 2.47 kg, determine the weight of each piece after cutting.

(Use the most appropriate units for your answers).

4 A strand of human DNA is 2.5 nanometers in diameter. A powerful electron microscope magnifies the strand 10 million times. Calculate the diameter of the strand after magnification.

5 Convert:

a 1720 mm^2 to cm^2 **b** 3.45 cm^3 to mm^3 **c** 0.0025 m^2 to cm^2 **d** 23 600 mm^2 to m^2.

6 The largest thermonuclear bomb tested was a Russian bomb with a yield of 50 megatons. In 2017, North Korea tested a weapon with a yield of 250 kilotons. Determine how many times larger the Russian bomb was than the North Korean bomb.

7 Write the ratio 15 : 24:

a in its simplest form **b** in the form 1 : n.

DP style Analysis and Approaches SL

8 A caterer supplies both plain and fancy cakes for a party.

a Out of the cakes in the box $\frac{7}{10}$ are plain. Write the ratio of plain to fancy cakes in the form n : 1.

b Give that the caterer supplies 44 more plain cakes than fancy cakes, write down and solve an equation for the total number of cakes supplied, x.

9 A model boat is made to a scale of 7 : 80.

Calculate the height (in cm) of the mast in the model if the full-sized boat has a mast that is 4 m high.

DP style Applications and interpretation SL

10 Bruce exchanges Australian dollars (AUD) to Indonesian Rupiah (IDR) at the airport.

If he receives 1 850 000 IDR for 200 AUD, calculate the value of 1 AUD in IDR.

Later, in a currency exchange kiosk, Bruce sees that the exchange rate is 9877 IDR to 1 AUD. Calculate how many more Rupiahs Bruce would have received if he had waited to change his money at this kiosk.

Higher Level

11 A new toothpaste tube contains 75 ml. This is 6.25% less than the old tube. How much toothpaste was in the old tube?

Modelling and investigation

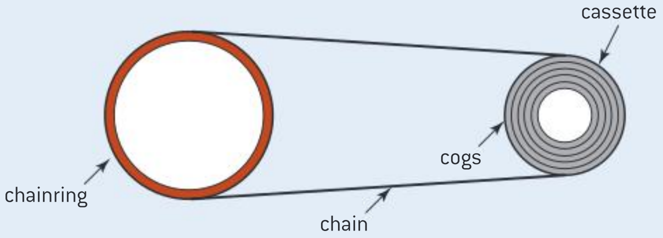
DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

Self-management: Managing time and tasks effectively

Bicycle gears



Bike riding is popular around the world.

Most bikes have both high gear ratios for cycling downhill fast and low gear ratios for cycling uphill slowly. Bikes have two sets of gearwheels:

- the larger front ones – the chainrings – are attached to the pedals
- the smaller rear ones – the cassette – are attached to the rear wheel of the bike.

The chain connects the two gearwheels.

On a standard set of gears, there are two chainrings and 11 cogs in the cassette. As you move from the larger to the smallest cog in the cassette, pedalling gets harder, but the speed at which you are cycling becomes faster. The smallest chainring is the easiest to pedal, but cycling speed is slower.

Each gearwheel is defined by its number of teeth.

Standard chainrings have 53 and 39 teeth.

The cogs in the cassette have 11, 12, 13, 14, 16, 18, 20, 22, 25, 28 and 32 teeth.

The gear ratio is defined as 'number of teeth in chainring : number of teeth in cassette'.

The hardest gear to cycle in is with the 53-tooth chainring at the front and the 11-tooth cog at the back. The gear ratio is 53 : 11 = 4.82 : 1. This means that one turn of the pedals will turn the rear wheel 4.82 times.

The easiest gear to cycle in is with the 39-tooth chainring at the front and the 32-tooth cog at the back. The gear ratio is 39 : 32 = 1.22 : 1. This gear is for cycling up steep hills.

Find the ratios for each of the 22 combinations of chainring and cog (to 3 s.f.).

'Cross-chaining' is using a combination of two big or two small gears, for example 53 and 32 or 39 and 11. Because there are many similar combinations, it is not necessary to cross-chain as it puts extra strain on the chain.

Put the ratios you found into order and find the combinations that will give you a full range of ratios and avoid cross-chaining by eliminating those that involve two big or two small gears.

A compact set of gears has chainrings with 48 teeth and 34 teeth combined with cogs that have 10, 11, 12, 13, 14, 16, 18, 20, 22, 25 and 28 teeth.

Calculate the ratios that are possible with this set of gears and compare them with the standard set.

FUNCTIONS

4

Graphing functions

Learning outcomes

In this chapter you will learn about:

- Graphing linear functions using technology
- Graphing quadratic functions using technology
- Mappings of the elements of one set to another. Illustration by means of sets of ordered pairs, tables, diagrams and graphs.
- Modelling with linear and quadratic functions

Key terms

- Graph
- Parabola
- Vertex
- Mapping
- Mapping diagram
- Domain
- Relation
- Image
- Range
- Function
- Vertical line test

4.1 Mappings

Suppose you have two sets, and every element of the first set is associated with an element of the other. This is called a **relation** between the two sets.

This table is an example of how the letters of the alphabet can be related onto the numbers 1 – 26.

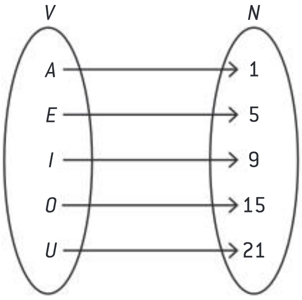
Internal link

In section 1.2, a **set** was defined as a collection of objects.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

Using this code, you would write I LOVE MATHEMATICS as 9 / 12 15 22 5 / 13 1 20 8 5 13 1 20 9 3 19.

This type of relation is often associated with a **mapping diagram**.



This diagram shows part of the relation above, but instead of the whole alphabet the relation only applies to the set of vowels, $V = \{A, E, I, O, U\}$.

The set $V = \{A, E, I, O, U\}$ is the called **domain** of the relation, and the set $N = \{1, 5, 9, 15, 21\}$ is called the **range**.

A relation gives you a set of pairs of objects: (A, 1), (E, 5), (I, 9), (O, 15), (U, 21) where the first coordinate in each pair is from the domain and the second coordinate is from the range.

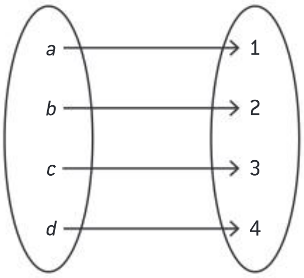
There are different types of relation:

Note

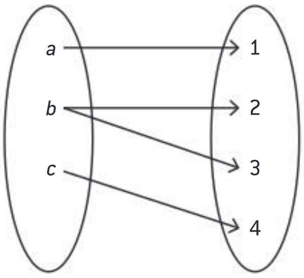
Can you list the coordinate pairs that each of these relations gives?

Note

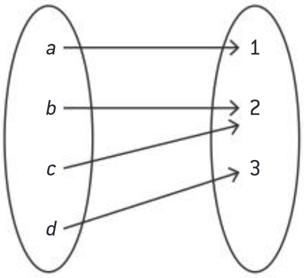
When you transform an element of a set, in this case using a relation, the result is called its **image**.



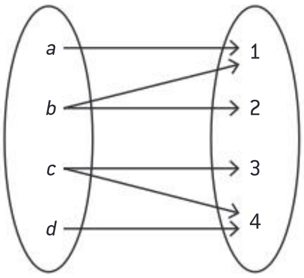
This type of relation is **one-to-one**: every element in the domain is related to one and only one element in the range.



This is **one-to-many**: elements of the domain can have more than one image.



This is **many-to-one**: more than one element from the domain can have the same image.



This is **many-to-many**: elements can have more than one image, and more than one element can have the same image.

A **mapping** is a special kind of relation. A relation is a mapping if every element of the domain maps to only one point in the range (a point in the range *can* have two or more points in the domain mapping to it, but every point in the domain *must* map to only one point in the range).

DP link

Although mappings can occur between any types of sets, the ones that you will be dealing with in the DP programme will mostly be mappings between sets of numbers.

Key point

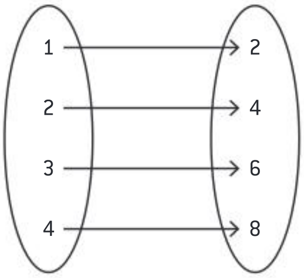
Only one-to-one and many-to-one relations are mappings because each point in the domain maps to one point in the range.

One-to-many and many-to-many relations are not mappings as one point in the domain can map to more than one point in the range.

Key point

The term **function** is often used to describe a mapping. If x is an element in the domain and it maps to y in the range, then we write $f: x \to y$ or $f(x) = y$.

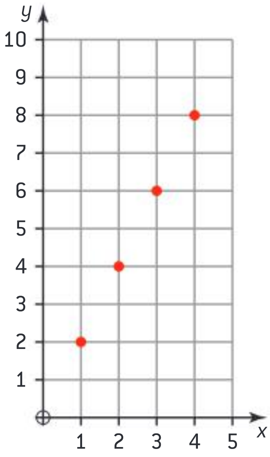
Here is a mapping diagram:



In this diagram, the domain is the set $\{1, 2, 3, 4\}$ and $f(1) = 2$, $f(2) = 4$, $f(3) = 6$, $f(4) = 8$. You can describe the mapping as $(1, 2)$, $(2, 4)$, $(3, 6)$, $(4, 8)$.

To write the function you could write $f: x \to 2x$, $f(x) = 2x$ or $y = 2x$.

Coordinate diagrams are generally used to show functions. You **plot** points in the domain on the x-axis and points in the range on the y-axis.



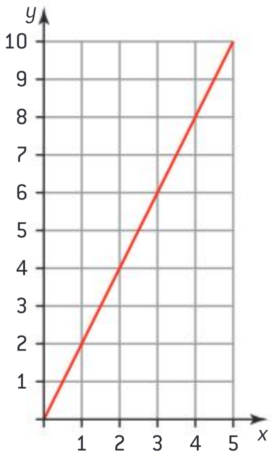
Command term

Plot means you should mark the position of points on a diagram.

Note

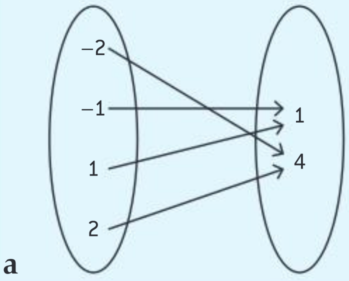
Because you can plot the graph of a function in this way, we sometimes write a function as $y = f(x)$ for some function involving x, such as $y = 2x + 1$ or $y = 4x^2$.

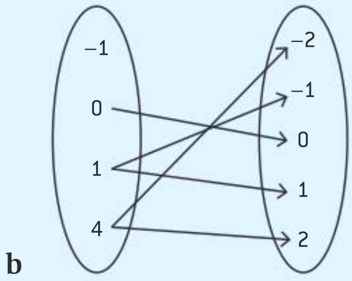
Here the function is a set of distinct points because the domain is a set of integers. If the domain was instead $\mathbb{R}$, then the graph of the function would be continuous.



Example 1

Determine which type of relation each of these represents, and hence state whether the relation is a function.





a function; many-to-one	Each element in the domain maps to only one point in the range.
b not a function; one-to-many and there is no image for –1.	1 maps to both –1 and 1, and 4 maps to both –2 and 2.

To draw the graph of a function you should calculate the coordinates of points related by the function, plot them and connect them with a smooth curve or a straight line.

Internal link

In sections 4.2 and 4.3 you will learn about the differences between functions whose graphs are straight lines and functions whose graphs are curves.

Example 2

Draw a graph of the function $y = 5 - 2x$

x	y
−2	9
0	5
4	−3

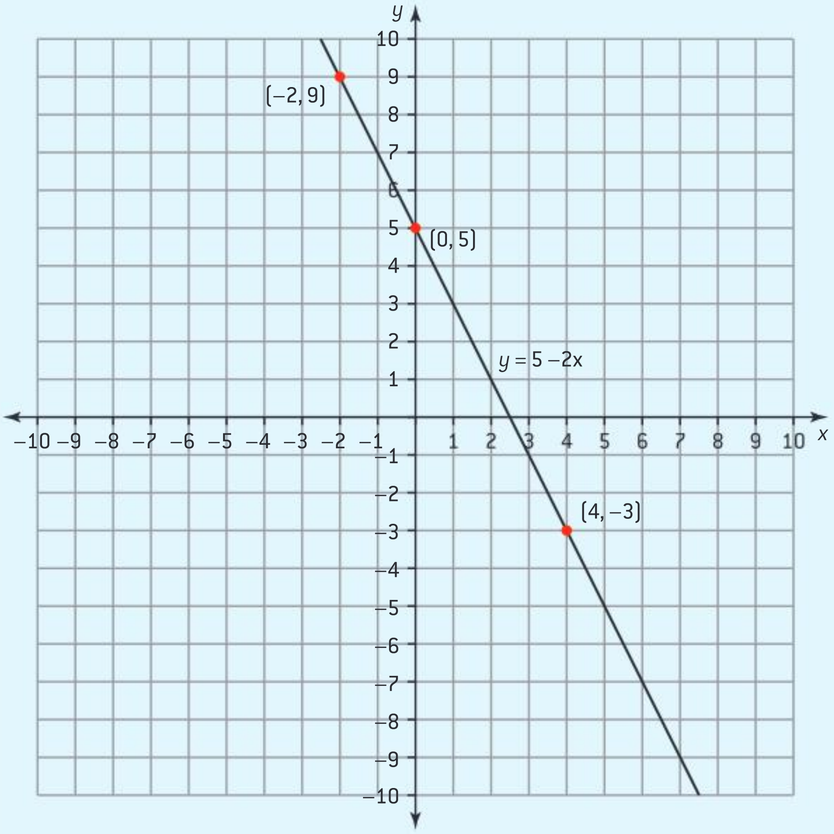
$(-2, 9), (0, 5), (4, -3)$

Choose a range of three values of x. Calculate y-coordinates and put them in a table of values.

$5 - 2(-2) = 9$

$5 - 2 \times 0 = 5$

$5 - 2 \times 4 = -3$



Plot the points on a graph.

Then draw a straight line through the points and label the line with its equation.

If the points given by a function do not lie in a straight line, you should make sure that you plot enough points to be able to draw a smooth curve through them. If possible, you should include any significant point, such as where the curve crosses the x- or y-axis.

Command term

Draw means you should represent, by means of a labelled, accurate graph, using a pencil. A ruler (straight edge) should be used for straight lines. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.

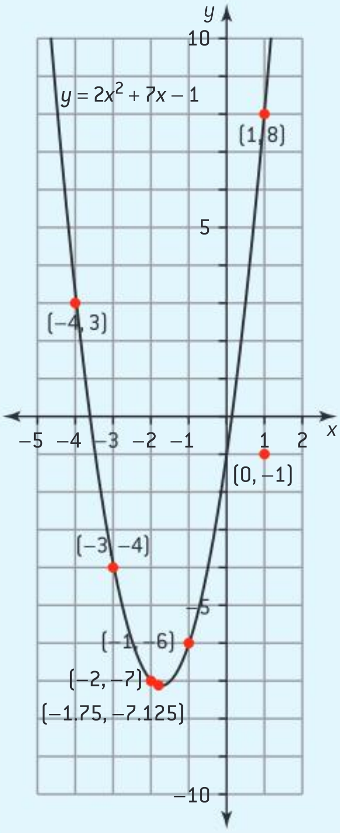
Example 3

Draw a graph of the function $y = 2x^2 + 7x - 1$.

x	y
−4	3
−3	−4
−2	−7
−1	−6
0	−1
1	8
−1.75	−7.125

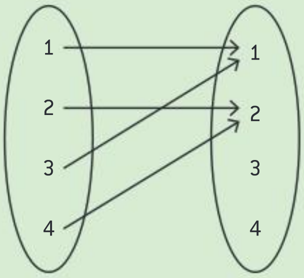
Choose a range of values of x. Calculate y-coordinates and put them in a table of values.

$2(-4)^2 + 7(-4) - 1 = 3$, etc.

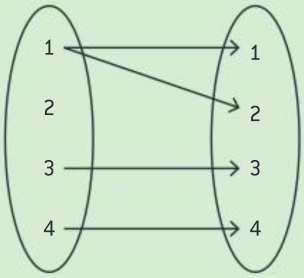
$(-4,3)$, $(-3,-4)$, $(-2,-7)$, $(-1,-6)$, $(0,-1)$, $(1,8)$, $(-1.75,-7.125)$	From the table, write the coordinates of the points.
	Plot the points on a graph. Then draw a smooth curve through the points and label it with its equation.

Exercise 4.1

1 Determine which of these diagrams show a function



a



b

2 Draw a graph of the function $y = 3x - 1$

3 Draw a graph of the function $y = x^2 - 3x - 4$

4.2 Graphing linear functions using technology

In section 4.1 you learned to draw the graph of a function by plotting points on paper. In this section you will look at functions whose graphs are straight lines.

Your GDC can plot the graph of a function for you. Your calculator is likely to either use the $y = \ldots$ notation, where you will enter the equations as Y1, Y2, etc, or the function notation where you enter equations as $f1(x)$, $f2(x)$, etc.

Some GDCs have an equation entry screen and others have drop-down area in a graph screen to enter the equations of functions.

DP link

In the DP course, you will learn how to calculate the coordinates of the vertex (the bottom point) of a graph like this one.

DP ready International-mindedness

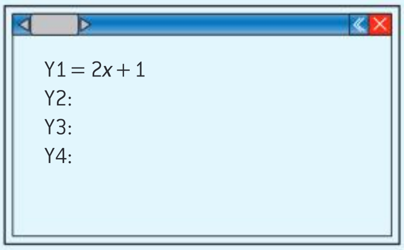
Coordinate geometry is often referred to as Cartesian geometry after one of its inventors. Descartes wrote his book, *la géométrie* in the early 17th century. In it he laid the foundations of modern analytic geometry. At around the same time Fermat wrote *ad locos planos et solidos isagoge* in which he independently produced many of the features of analytic geometry. Analytic geometry combined geometry and algebra as a single subject and provided the basis for later mathematical breakthroughs, like Newton and Leibniz developing the differential calculus (which you will learn about in both the MAA and MAI courses.)

Key point

A **linear function** is a function whose graph is a straight line. It can be written in the form $y = mx + c$ where m and c are numbers.

Example 4

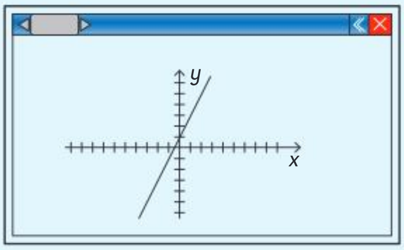
Graph the function $y = 2x + 1$ with your GDC.



Y1 = 2*x* + 1
Y2:
Y3:
Y4:

Open the equation entry screen, type $2x + 1$ and press Enter (or EXE).

To type x, you may have to press a key labelled something like X, T, θ.



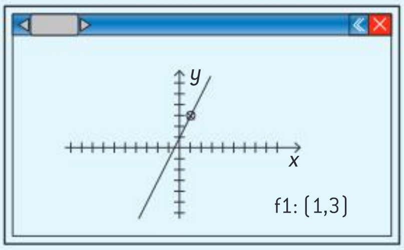
Switch to the graph screen.

The GDC displays the graph of the function $y = 2x + 1$ with the default axes.

Using the GDC you can trace points on the line. The GDC has arrow keys and you can use these to move the cursor along the line to display the coordinates of the point. You can also type the value of an x-coordinate.

Example 5

Find the coordinates of the point on $y = 2x + 1$ where $x = 1$.



f1: (1,3)

Using the trace function on your GDC, type the x-coordinate, 1.

The GDC displays the coordinates of a point on the line $(1, 2)$.

Key point

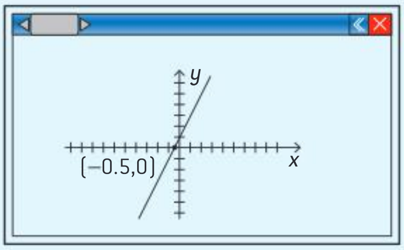
The points where a line crosses the axes are the x- and y-intercepts.

GDCs will find the x-intercept with a function called zero or root. The x-intercept is the y-value when $x = 0$, so it is the point at which the graph crosses the y-axis.

Some GDCs also have a function that will find the y-intercept. If this function is missing then you will be able to trace a point where $x = 0$.

Example 6

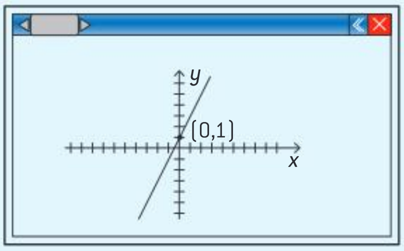
Find the x- and y-intercepts of the function $y = 2x + 1$.



(−0.5,0)

Use the zero or root function to find the x-intercept. Follow any instructions on the screen.

The GDC displays the coordinates of the point $(-0.5, 0)$.



Using the trace function on your GDC, type the x-coordinate, 0, or use the y-intercept function.

The GDC displays the coordinates of a point on the line (0,1).

In DP examinations, you will also have to **sketch** the graphs of functions.

Sketching is different from *drawing* an accurate plotted graph, but you should avoid making your sketches too unclear. You do not need to use graph paper for a sketch but it is advisable to use a pencil and a ruler. You should **label** axes, but you do not need to give a full scale. You should label key points (such as where the graph passes through the axes) with their coordinates.

Command term

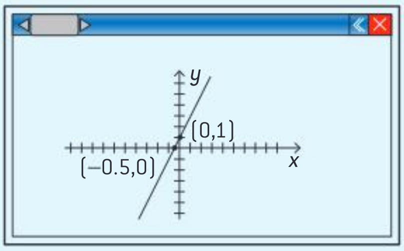
Sketch a graph means you should represent by means of a graph (labelled as appropriate). The sketch should give a general idea of the required shape or relationship, and should include relevant features.

Draw a graph means you must correctly plot coordinates and join them with a straight line or smooth curve. You did this in section 4.1

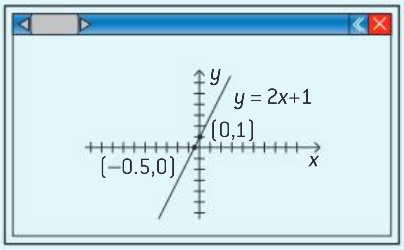
Label means you should add labels to the diagram or graph. This might mean labels for the x- and y-axes, or the coordinates of points on the graph.

Example 7

Sketch the graph of the function $y = 2x + 1$.



Use the GDC to graph the function $y = 2x + 1$ and find the x- and y-intercepts. These are the key points.



Draw and label the axes. A scale is not really necessary as the intercepts are labelled.

Label the intercepts with their coordinates.

Sketch the line using a ruler and passing through the labelled intercepts.

Label the equation of the function.

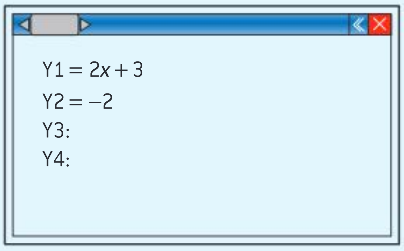
Internal link

In section 2.5a you learned how to solve linear equations algebraically.

Example 8 will show how to solve a linear equation graphically, using your GDC.

Example 8

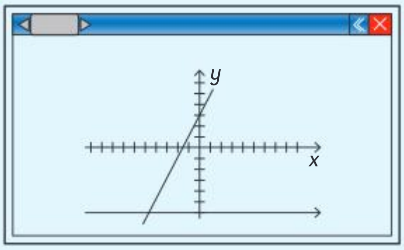
Solving the equation $2x + 3 = -6$ graphically, using your GDC.



Y1 = 2x + 3
Y2 = −2
Y3:
Y4:

Solving the equation $2x + 3 = -6$ is equivalent to finding the point on the graph of $y = 2x + 3$ where $y = -6$.

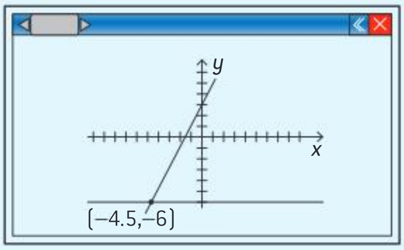
Enter the functions $y = 2x + 3$ and $y = -6$.



Switch to the graph screen.

Your GDC will display two straight line graphs.

You can find the solution of $2x + 3 = -6$ by finding the point of intersection of $y = 2x + 3$ and $y = -6$.



Your GDC will tell you the coordinates of this point, which is $(-4.5, -6)$.

The solution to $2x + 3 = -6$ is $x = -4.5$.

DP ready: MAA & MAI

The MAA course focuses more on sketching graphs by analysing the equation of a function and finding key points by which to sketch a graph. The MAI course focuses more on sketching graphs by first plotting them on your GDC.

The GDC graphs linear *explicit* functions – that is, functions that start with '$y =$'. For example, $y = 3x - 4$ or $y = 2(x + 1)$. You cannot enter an *implicit* equation such as $2x + 3y = 6$ unless you first rearrange it to make y the subject.

Internal link

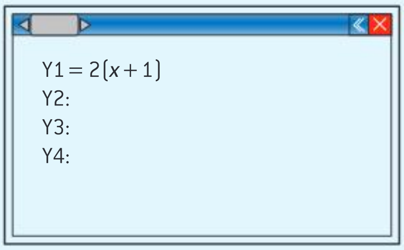
See section 2.5c to recap rearranging an equation.

Example 9

Graph these equations on your GDC:

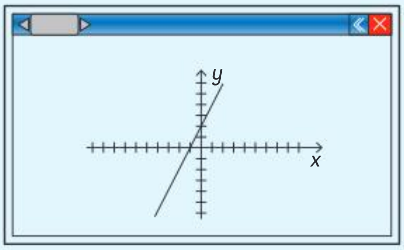
a $y = 2(x + 1)$ **b** $2x + 3y = 6$

a



Y1 = 2(x + 1)
Y2:
Y3:
Y4:

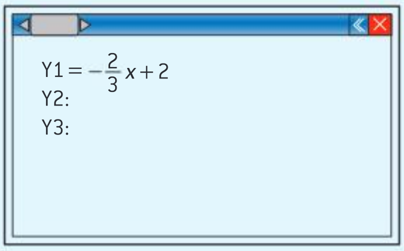
Enter the function $y = 2(x + 1)$



Switch to the graph screen.

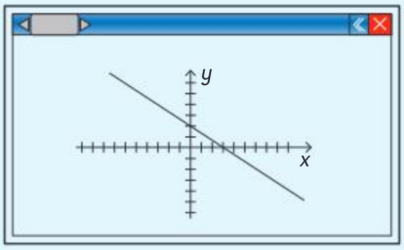
Your GDC will display the graph of the function.

b



Y1 = $-\frac{2}{3}x + 2$
Y2:
Y3:

$$\begin{aligned} 2x+3y&=6 \\ 3y&=-2x+6 \\ y&=-\frac{2}{3}x+2 \end{aligned}$$



Enter the function $y=-\frac{2}{3}x+2$

Switch to the graph screen.

Your GDC will display the graph of the function.

Look at the pair of simultaneous equations $\begin{matrix}5x+3y=29\\2x+3y=17\end{matrix}$.

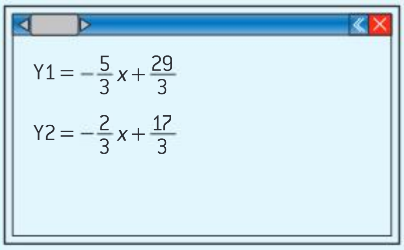
Both equations are written as implicit functions. Writing these as explicit equations (where y is the subject) allows you to solve them using your GDC by finding the intersection of the two lines.

Internal link

In section 2.9, you learned to solve simultaneous equations.

Example 10

Solve the equations $\begin{matrix}5x+3y=29\\2x+3y=17\end{matrix}$ graphically, using your GDC.



$Y1=-\frac{5}{3}x+\frac{29}{3}$

$Y2=-\frac{2}{3}x+\frac{17}{3}$

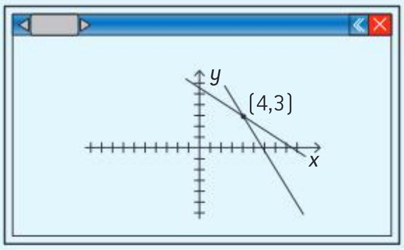
Rearrange the equations to make y the subject.

$$y=-\frac{5}{3}x+\frac{29}{3}$$

$$y=-\frac{2}{3}x+\frac{17}{3}$$

Enter the functions into your GDC

Switch to the graph screen.



The GDC shows two straight lines.

Use the intersection function of your GDC to find the point where the lines cross.

$x=4$ and $y=3$

The intersection point is (4, 3) and so the solution is $x=4$ and $y=3$.

Exercise 4.2

1 Use your GDC to graph these functions, find the x- and y-intercepts and sketch their graphs.

a $y=x-4$ **b** $y=\frac{1}{2}x+3$ **c** $y=3-2x$

2 Use your GDC to plot $y=\frac{2}{3}x+\frac{4}{5}$ and find the point on $y=\frac{2}{3}x+\frac{4}{5}$ where $x=1.75$.

3 Solve these equations graphically using your GDC:

a $3x-5=4$ **b** $4x-3=2$ **c** $3(x-4)=2$

4 Solve the simultaneous equations $\begin{matrix}3x+2y=3\\4x+4y=12\end{matrix}$ graphically, using your GDC.

4.3 Graphing quadratic functions using technology

Key point

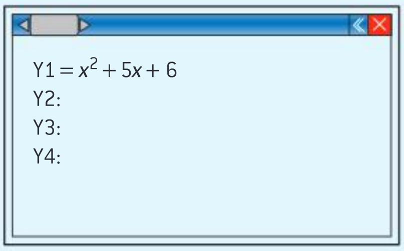
A **quadratic function** is in the form $y=ax^2+bx+c$ or, in factorized form, $y=a(x-p)(x-q)$.

Internal link

You studied quadratic expressions in section 2.7. A quadratic expression is one where the highest power of x is 2, i.e. it contains a term in x^2.

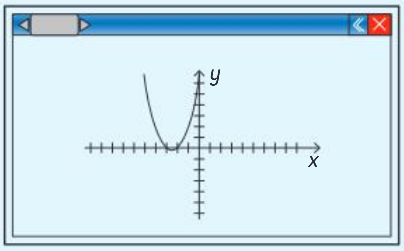
Example 11

Graph the function $y = x^2 + 5x + 6$ with your GDC.



$Y1 = x^2 + 5x + 6$
Y2:
Y3:
Y4:

Open the equation entry screen, type $x^2 + 5x + 6$



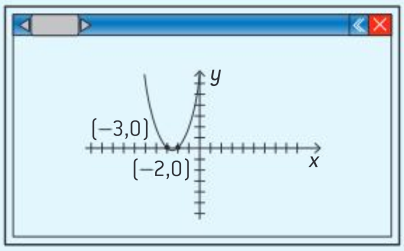
Switch to the graph screen.

The GDC displays the graph of the function $y = x^2 + 5x + 6$ with the default axes.

The shape of the quadratic curve is a **parabola**. It is a smooth curve and symmetrical about a vertical line called an axis. Parabolas can be ∪-shaped or ∩-shaped.

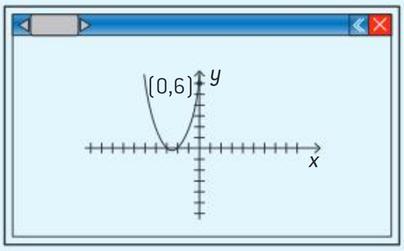
Example 12

Find the x- and y-intercepts of the function $y = x^2 + 5x + 6$.



Use the zero or root function to find the x-intercepts. Follow any instructions on the screen.

The GDC displays the coordinates of the points $(-3, 0)$, $(-2, 0)$.



Using the trace function on your GDC, type the x-coordinate, 0, or use the y-intercept function.

The GDC displays the coordinates of a point on the line $(0, 6)$.

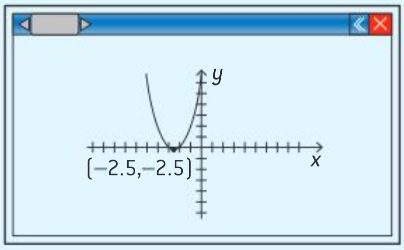
Note

If you are asked to find the intercept, then the answer is a single number. In the example above, the y-intercept is 6. If you are asked for the coordinates of the y-intercept, you need to write (0,6). Generally, exam mark schemes will accept the coordinates if asked for the intercepts, but not vice versa.

There is another important point on the quadratic curve: the **vertex**, which is its lowest or highest point on the curve. To find this point you can use the minimum or maximum function on your GDC.

Example 13

Find the vertex of the function $y = x^2 + 5x + 6$.



Use the minimum function of the GDC to find the vertex.

The GDC displays the coordinates of the point.

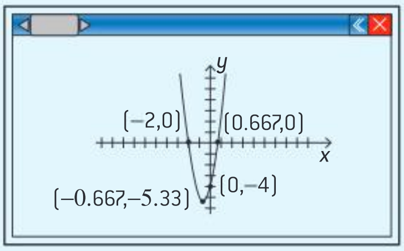
When you sketch a quadratic function, you should show and label the key features: its intercepts and vertex.

Example 14

Sketch the graph of the function

a $y = 3x^2 + 4x - 4$ **b** $y = 2x^2 + 4x + 3$ **c** $y = 4x^2 - 12x + 9$

a

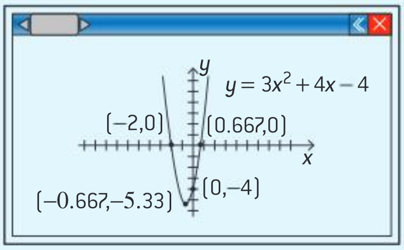


Use the GDC to graph the function $y = 3x^2 + 4x - 4$.

Find the x- and y-intercepts and the vertex.

These are the key points.

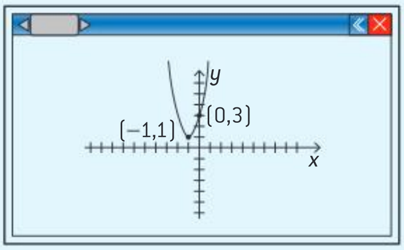
Draw and label the axes. A scale is not really necessary as points are labelled.



Sketch the curve and label the intercepts and the vertex with their coordinates.

Label the equation of the function.

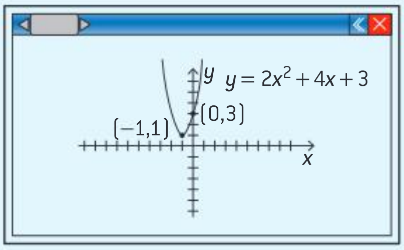
b



Use the GDC to graph the function $y = 2x^2 + 4x + 3$.

Find the y-intercept and the vertex. There are no x-intercepts.

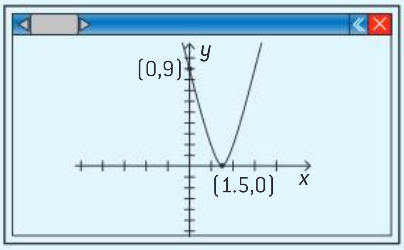
Draw and label the axes. A scale is not really necessary as points are labelled.



Sketch the curve and label the intercept and the vertex with their coordinates.

Label the equation of the function.

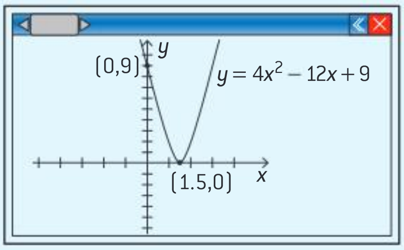
c



Use the GDC to graph the function $y = 4x^2 - 12x + 9$.

Find the y-intercept and the vertex. The vertex is on the x-axis and so it is the x-intercept.

Draw and label the axes. A scale is not really necessary as points are labelled.



Sketch the curve and label the intercept and the vertex with their coordinates.

Label the equation of the function.

DP ready: Analysis and approaches

This investigation is in the style of those from Analysis and approaches.

Command term

Explain means you should give a detailed account, including reasons or causes.

Investigation 4.1

1 Sketch the graphs of $y = x^2 - 3x + c$ where $c = -1, 0, 1, 2, 3, 4$. Label the y-intercept and the vertex of the curves.

- **a** What do you notice about the y-intercept and the value of c? **Explain** why this pattern exists.
- **b** Describe the pattern that you notice for the vertex of the curves and the value of c.

2 Sketch the graphs of $y = x^2 + bx + 1$ where $b = -4, -3, -2, -1, 0, 1, 2, 3, 4$. Label the vertices of these curves.

What do you notice about the x-coordinate of the vertex and the value of b? Predict where the vertex of $y = x^2 + 6x + 6$ will be and confirm your prediction with your GDC.

3 For this part of the investigation, you may need to adjust the viewing window of your GDC. You can adjust the scales by changing the maximum and minimum values of the x- and y-axes.

Sketch the graphs of $y = ax^2 + 6x + 6$ where $a = 1, 2, 3$. Label the vertices of these curves. What do you notice about the x-coordinate compared to the result you obtained for question 2? Predict what the x-coordinate of the vertex of $y = 5x^2 + 6x + 6$ will be and confirm your prediction with your GDC.

DP ready Theory of knowledge

In an advanced mathematical textbook published in 1960, intended for students going on to study mathematics at university, there was a whole chapter devoted to techniques for manipulating quadratic functions and solving quadratic equations. The IB first approved the use of calculators in examinations in the 1980s, and in the 1990s they permitted some use of graphical display calculators. How do you think that this change has influenced what is in mathematics syllabuses over recent decades? Do you think that mathematicians need to be good at solving things by hand when there is technology that can perform the same functions at the press of a button?

Exercise 4.3

1 Use your GDC to graph these functions, find the x- and y-intercepts and draw a sketch.

a $y = \frac{1}{2}x^2 - 2x - 3$ **b** $y = 9x^2 - 4$ **c** $y = \frac{3}{4}(x+1)^2$

2 Use your GDC to find the point on $y = x^2 + x + 1$ where $x = 1.5$

3 Solve these equations graphically using your GDC:

a $\frac{1}{2}x^2 = 4.5$ **b** $x^2 + 2x = 3$ **c** $x^2 + 6x + 10 = 1$

4 Find the vertex of $y = x^2 - 5x + 4$ using your GDC.

DP style Analysis and Approaches SL

5 A quadratic curve $y = ax^2 + bx + c$ has a y-intercept of 9 and a vertex at $(2,1)$.

- **a** State the value of c.
- **b** Sketch the curve and write down the coordinates of another point on the curve.
- **c** Use the points the curve passes through to write down two linear equations in a and b.
- **d** Solve these two equations to find the equation of the curve.

4.4 Linear and quadratic models

You can model many situations in science, economics, etc. by linear functions. You can use a graph of the function, either drawn by hand or with a GDC, to calculate values of the function.

Example 15

The final velocity, v, of an object, accelerating at a rate of a is given by the formula $v = u + at$ where u is the initial velocity and t is the time.

If $u = 33$ ms^{-1} and $a = -10$ ms^{-2}, use the function $v = 33 - 10t$ to model the motion of a cricket ball. You can enter this into your GDC as $y = 33 - 10x$, where y takes the place of v and x takes the place of t.

Use your GDC to graph the function $v = 33 - 10t$.

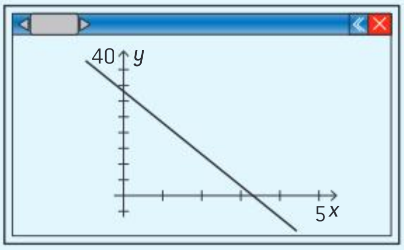
a Find the value of v when $t = 2$ s.

b Find the value of v when $t = 4$ s. Comment on your result.

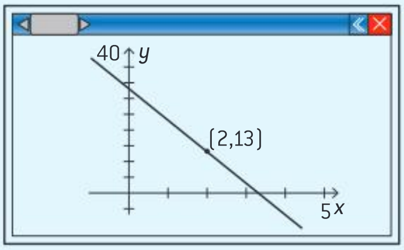
c Find the value of t when $v = 0$.

d Find the value of t when $v = 10$ ms^{-1}.

a



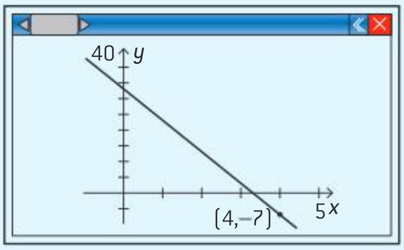
You will need to adjust the window to view the function to $-1 \le x \le 5$ and $-10 \le y \le 40$.



Use the trace function to find y when $x = 2$.

$v = 13$ ms^{-1}

b

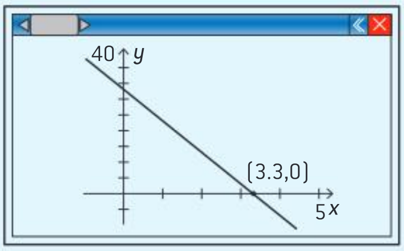


Use the trace function to find y when $x = 4$.

$v = -7$ ms^{-1}. The direction of the ball's motion is downward.

In the DP course you will learn that the sign of velocity tells you whether the object is moving in the positive or the negative direction.

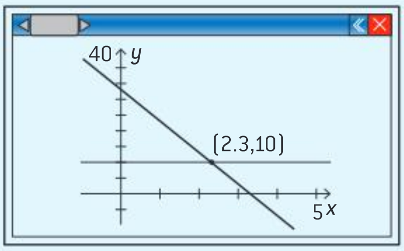
c



Use the zero (or root) function.

$t = 3.3$ s.

d



Draw the line $y = 10$ and find the intersection.

$t = 2.3$ s.

You can also model situations in the same way with quadratic functions.

Example 16

The height, s, of an object, accelerating at a rate of a is given by the formula $s = ut + \frac{1}{2}at^2$ where u is the initial velocity and t is the time.

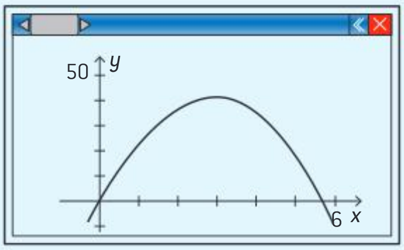
A football is kicked vertically upward with $u = 28\ \text{ms}^{-1}$ and $a = -10\ \text{ms}^{-2}$.

a Find the value of s when $t = 2$ s.

b Find the value of t when $s = 0$.

c Find the value of t when $s = 25$ m.

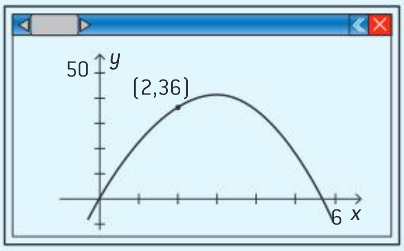
d Find the maximum height of the football.



Enter the function $y = 28x - 5x^2$ into your GDC to model the motion of a football, where y takes the place of s and x takes the place of t. Use your GDC to graph the function.

You will need to adjust the window to view the function to $0 \le x \le 6$ and $0 \le y \le 50$.

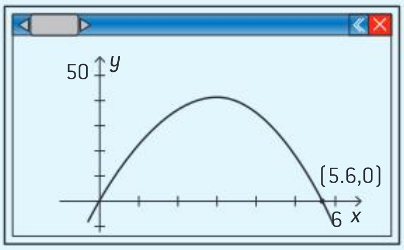
a



$s = 36$ m

Use the trace function to find y when $x = 2$.

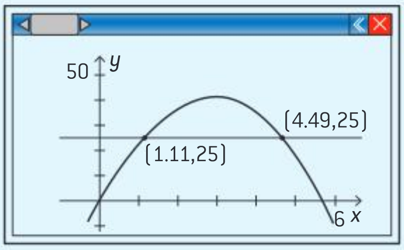
b



$t = 0$ s and 5.6 s.

Use the zero (or root) function.

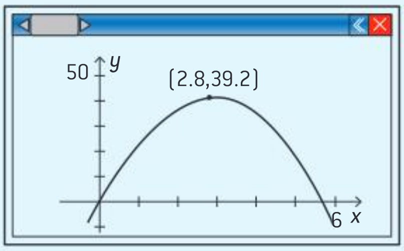
c



$t = 1.11$ s or 4.49 s.

Draw the line $y = 25$ and find the intersection.

d



Maximum $s = 39.2$ m when $t = 2.8$ s.

Use the maximum function to find the vertex of the curve.

Investigation 4.2

Use your GDC to complete this table.

equation	$y = 2x - 4$	$y = 2x - 3$	$y = 2x - 2$	$y = 2x - 1$	$y = 2x$	$y = 2x + 1$	$y = 2x + 2$	$y = 2x + 3$	$y = 2x + 4$
y-intercept									

Comment on the pattern that you see.

Confirm your results by examining the y-intercepts of some other linear functions.

The **gradient** of a line is a measure of how steep it is. To find the gradient with a GDC you will need to use a function referred to as dy/dx. (On some GDCs you will need to use the *derivative* function). When using this function, you choose a point on the line and the GDC will give the value of the gradient. The gradient of a straight line will be the same for every point on the line.

Use you GDC to complete this table.

equation	$y = -\,4x + 1$	$y = -\,3x + 1$	$y = -\,2x + 1$	$y = -\,x + 1$	$y = 1$	$y = x + 1$	$y = 2x + 1$	$y = 3x + 1$	$y = 4x + 1$
gradient									

Comment on the pattern that you see.

Confirm your results by examining the gradients of some other linear functions.

Use you GDC to complete this table.

equation	$y = x^2 - 3$	$y = 2x^2 - x - 2$	$y = 2x^2 + 2x - 1$	$y = x^2 + 4x$	$y = 3x^2 - 3x + 1$	$y = x^2 + 3x + 2$	$y = x^2 + x + 3$
y-intercept							

Comment on the pattern that you see.

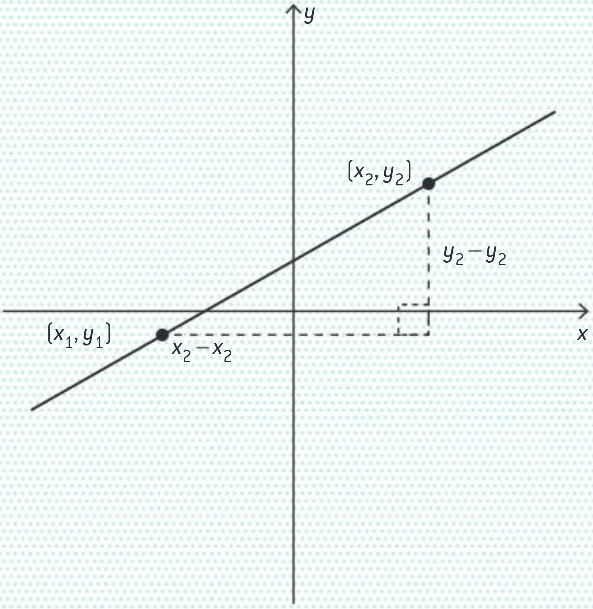
Confirm your results by examining the y-intercepts of some other quadratic functions.

Higher Level

4.5 Finding the equation for a linear model

As seen in the investigation, in a linear model of the form $y = mx + c$, m represents the gradient or slope of the graph of the model. The gradient is a measure of the rate at which y changes with respect to x. It is the amount that y increases or decreases as x increases by one unit. Finding m will enable us to find the equation for a linear model. To find m you need to know two pairs of x and y values.

If two points on a straight line are (x_1, y_1) and (x_2, y_2), then the gradient of the line (or the rate of increase of y with respect to x) is given by $\frac{y_2 - y_1}{x_2 - x_1}$.



Example 17

The temperature on a beach (T) between the times of 0700 and 1400 can be modelled by the function $T = mt + c$ where t is the time in hours after midnight. At 0700 the temperature is 12°C and at 1200 the temperature is 24°C.

a Find the rate of increase of the temperature in degrees per hour.

b Find the values of m and c for the model.

c Hence predict the temperature at 1400.

a $\frac{24-12}{5} = 2.4$°C/hour	The rate of increase will be the total increase divided by the length of time between the two points.
b $m = 2.4$ $12 = 2.4 \times 7 + c$ $c = -4.8$	The value of m is simply the rate of increase of temperature. As the equation must be satisfied by both the sets of values given, either can be substituted to find c.
c $T = 2.4t - 4.8$ $T = 2.4 \times 14 - 4.8 = 28.8$	The second point can be used as a check $2.4 \times 12 - 4.8 = 24$

DP ready: Applications and interpretation

The following questions are more in the style of those you would find on the MAI paper.

Exercise 4.4

1 The quantity of items sold, q and their selling price, s are connected by the formula $q = -20s + 1200$.

Use your GDC to graph the function for $0 \le s \le 25$ and $0 \le q \le 1300$.

Find:

a The quantity sold when the selling price is 20.

b The selling price when 1000 items are sold.

2 The profit, $P(x)$ made when selling x television sets is modelled by the function $P(x) = -20x^2 + 1400x - 12000$.

Use your GDC to graph the function for $0 \le x \le 100$ and $0 \le y \le 15000$.

From the graph find:

a The number of television sets that must be sold before the company break even.

b The maximum profit and the number of television sets that must be sold to achieve this.

Note

Break even means to make zero profit.

Chapter summary

- ☐ A function is a mapping for which there is only one value of y for each value of x.
- ☐ A **linear function** is a function whose graph is a straight line. It is in the form $y = mx + c$.
- ☐ The points where a line crosses the axes are the x- and y-**intercepts**.
- ☐ A **quadratic function** is in the form $y = ax^2 + bx + c$ or, in factorized form, $y = a(x-p)(x-q)$. Its graph is a parabola.

Chapter 4 test

1 Use your GDC to graph the function $y = 2(2 - x)$ and find the coordinates of the x- and y-intercepts.

2 Use your GDC to graph the function $y = 2.05x - 3.65$ and find the coordinates of the point on the line where $x = 3.25$.

3 Use your GDC to draw the graph of $y = 5 - \frac{1}{2}x$ and **hence** solve the equation $5 - \frac{1}{2}x = 2$.

4 Sketch the graph of $y = \frac{1}{2}x^2 + 2x - 3$, showing the coordinates of the intercepts and vertex.

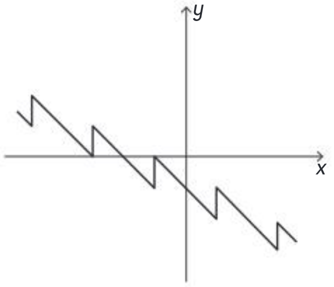
Command term

Hence means you should use the preceding work to obtain the required result.

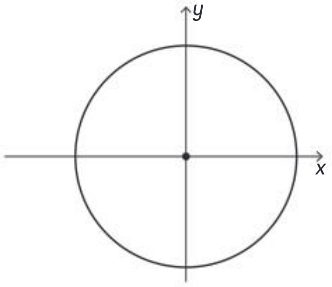
5 Use your GDC to draw the graph of $y = x^2 - 3x - 4$ and hence solve the equation $x^2 - 3x - 4 = 2$.

6 Use your GDC to draw the graph of $y = x^2 + 4x - 1$. Find the coordinates of the x- and y-intercepts and the vertex. Show that the vertical line through the vertex is equidistant from the intercepts.

7 Which of these graphs are functions?



a



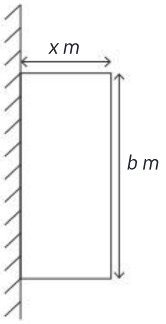
b

DP style **Analysis and Approaches SL**

8 Without using your GDC, draw the graph of the function $y = (x+3)(2-x)$ for $-4 \le x \le 4$, showing clearly the values of the x- and y-intercepts and the coordinates of the vertex.

DP style **Applications and Interpretation SL**

9 A total of 25 m of fencing is being used to make a rectangular enclosure in a field to contain some sheep. One side of the enclosure is a straight wall and the other three sides are made using the fencing, as shown in the diagram.



If one side of the enclosure is x m long, write down the length, b cm of the other side in terms of x and hence find the area of the enclosure in terms of x.

Graph the area function using your GDC and find the maximum area for the enclosure.

Modelling and investigation

DP ready **Approaches to learning**

Critical thinking: Analyzing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

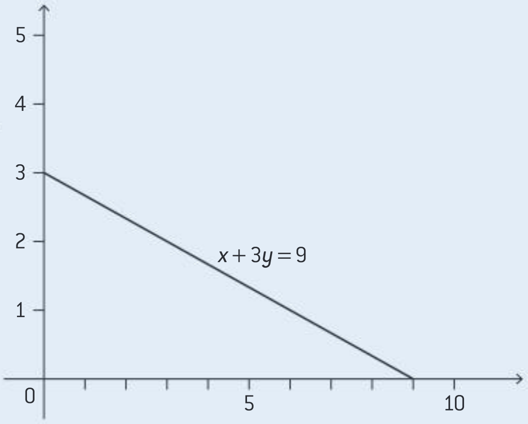
During the second world war, a Russian economist Leonid Kantorovich developed linear programming as a method for achieving the best result in a situation where the constraints can be represented as linear functions. Linear inequalities are used to represent any restrictions.

An inequality can also have two variables, x and y, such as $x + 3y < 9$. Instead of the solution being a portion of the number line, the solution of this inequality is a region of the Cartesian plane.

First, you must look at the points where $x + 3y = 9$. This is a straight line. You can draw this by finding several points on the line that satisfy its equation.

On one side of this line $x + 3y < 9$ and on the other $x + 3y > 9$.

To decide which side is which, you can choose one point, for example the point $(0,0)$. At this point $x = 0$ and $y = 0$. Substituting this into the expression $x + 3y$, $0 + 3 \times 0 = 0$ and $0 < 9$, so $(0,0)$ is in the region $x + 3y < 9$. The required region is to the left of the line. This can be verified by trying other points.



Consider the following problem which will be solved by linear programming.

You are responsible for the transportation of supplies to a remote area that has been struck by an earthquake. Supplies can be transported by plane or truck and you have a budget for getting them to the area.

Let x be the number of trips by plane and y the number of trips by truck.

1 The cost of a plane is \$1000 per trip and the cost of a truck is \$500 per trip. Your budget is \$62 500. Show that $2x + y \le 125$.

2 There are 5 planes and 15 trucks available.

Planes can make 2 trips per day

Trucks will take 2 days per trip (1 day to reach the site and a second to return).

Write inequalities in the form $x \ge a$, $x \le b$, $y \ge c$ and $y \le d$ for the number of trips by plane and truck that are possible in 6 days.

3 On graph paper, or using an online graphing package, draw the lines representing the equations

$2x + y = 125$, $x = a$, $x = b$, $y = c$ and $y = d$

Indicate on your graph the region containing the values of x and y which satisfy all the inequalities. This is known as the feasible region.

4 Planes can deliver 1 tonne of supplies per trip. Trucks can deliver 2 tonnes of supplies per trip.

Let A be the amount of supplies that can be delivered to the area in 6 days. Write down an expression for A in terms of x and y.

The fundamental theorem of linear programming states that the maximum or minimum value of any linear function will occur at one of the vertices of the feasible region.

Use the theorem to find the maximum amount of supplies that can be delivered in the first six days and the number of planes and trucks that should be hired.

5 Geometry

Learning outcomes

In this chapter you will learn about:

- Geometric concepts: point, line, plane, angle
- Angle measurement in degrees.
- The triangle sum theorem
- Properties of triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapezoids; compound shapes
- Three-figure bearings, compass directions
- Simple geometric transformations: translation, reflection, rotation, enlargement

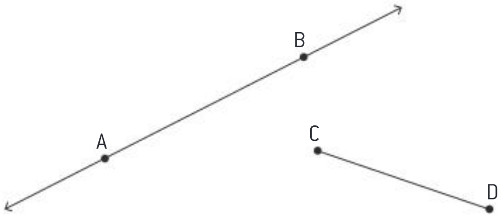
Key terms

- Point
- Line
- Line segment
- Parallel
- Angle
- Perpendicular
- Supplementary
- Acute
- Obtuse
- Vertically opposite
- Corresponding
- Alternate
- Co-interior
- Complementary
- Bisect
- Reflection
- Rotation
- Translation
- Enlargement
- Vector
- Stretch

5.1 Points, lines, planes and angles

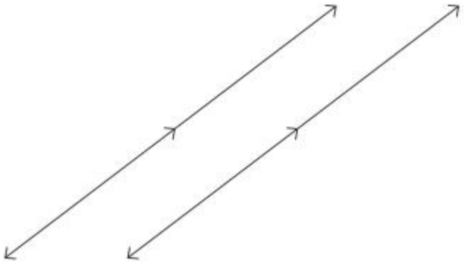
The idea of a **point** is fundamental in geometry. A point is a location in space. It is dimensionless, that is it has no length, width or depth. A point is shown by a dot, •.

A straight **line** connects points and has one dimension only, length. It has no depth or breadth. A line through two points continues infinitely. A line which has two endpoints is called a **line segment**.



The line through A and B is (AB) and the line segment joining C and D is [CD].

If two lines do not meet, then they are **parallel**. Arrows are used to show that two lines are parallel.



DP ready **Theory of knowledge**

Mathematical ideas are based on axioms, or 'self-evident truths' and are developed logically from them. The first such system was the geometry of Euclid in ancient Greece. The axioms of plane geometry defined the point, line, plane and angle, which are not defined in terms of any other objects. Euclidean geometry has changed little in the last 2000 years.

Key point

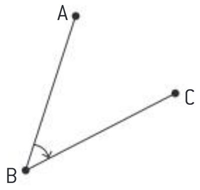
A point is a dimensionless location in space.

A line is one-dimensional and extends infinitely in both directions.

A line segment is a line with two endpoints.

Two lines are parallel if they are the same distance apart and never touch.

The amount of turn between two line segments around the point where they meet is the **angle** between them.

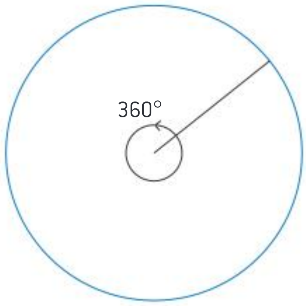


This angle can be described as $A\hat{B}C$ or, if the description is clear because there are no other angles at the point B which could cause confusion, as $\hat{B}$.

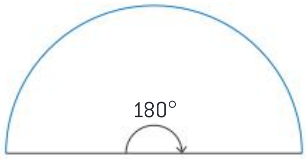
Angles are measured in degrees. A full turn that rotates all the way round and back to the start is 360°.

DP link

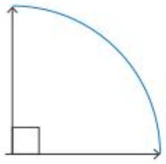
DP students taking MAA SL and HL and MAI HL will learn about an alternative measure of angles known as radians. 1 radian measures out an arc length on a circle equal to its radius.



The angle on a straight line is known as a straight angle. The measure of a straight angle is 180°.

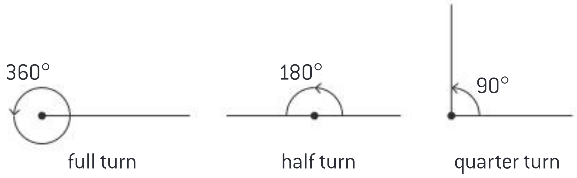


Half the angle in a straight line is a right-angle. You show that an angle is a right angle with a symbol like a box in the corner.

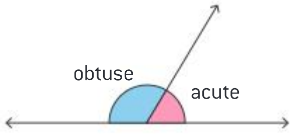


A right angle is 90°. Two lines at right-angles are called **perpendicular.**

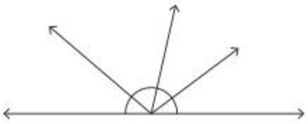
A full turn is 360°, a straight angle (180°) is a half-turn and a right-angle (90°) is a quarter-turn.



If you split a straight angle in two, then the angles formed by splitting it are called **supplementary**. Supplementary angles add up to 180°.



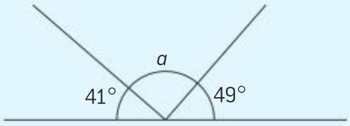
One of these angles is **obtuse** and the other is **acute**. An obtuse angle is between 90° and 180° and an acute angle is less than 90°.



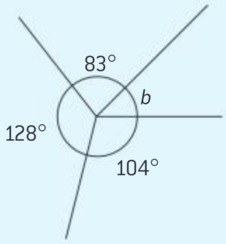
Any group of angles on a straight line will add up to 180°.

Example 1

a Calculate the angle marked a.



b Calculate the angle marked b.



a $41 + a + 49 = 180$

$90 + a = 180$

$a = 90°$

A straight angle is 180°.

b $83 + 128 + 104 + b = 360$

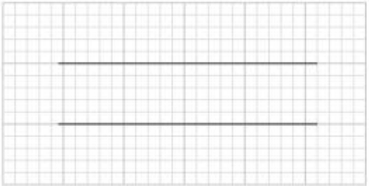
$315 + b = 360$

$b = 45°$

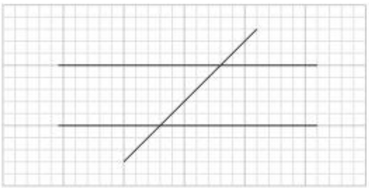
The full turn is 360°.

Investigation 5.1

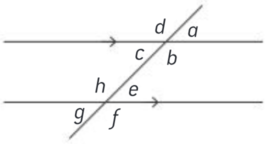
1 On a sheet of graph paper (or squared paper), draw two parallel lines.



2 Draw another line that crosses the two parallel lines.



3 For reference, label the angles as shown.



4 Using a protractor, measure all the angles.

5 What do you notice about the angles a and b?

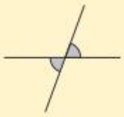
What do you notice about the angles b and c?

What can you deduce about the angles a and c? Try to explain why this follows from the first two observations you made.

What do you notice about b and d? Give a reason.

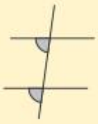
Key point

When two lines cross

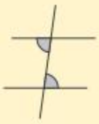


Vertically opposite angles are equal.

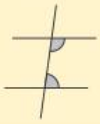
When a line crosses two parallel lines:



Corresponding angles are equal.



Alternate angles are equal.



Co-interior angles are supplementary.

6 Copy and complete the following, writing your answers in terms of angle *a* and giving reasons.

angle b = ……… (because ……………………….)

angle c = …….. (because ……………………….)

angle d = …….. (because ……………………….)

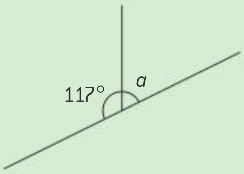
7 What do you notice about angles a and e?

8 What do you notice about angles c and h?

Exercise 5.1

1 Find the angles marked with letters.

i



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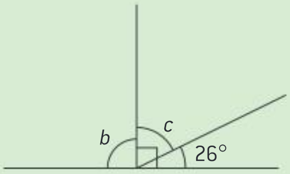

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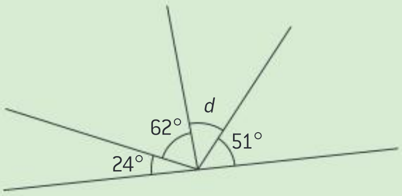

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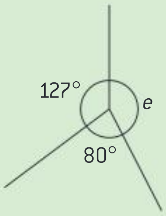
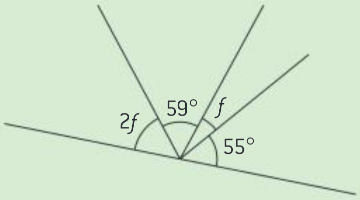


v



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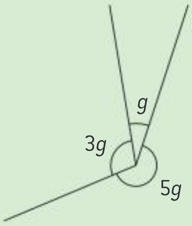
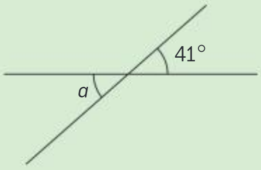


2 **i**



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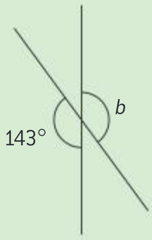

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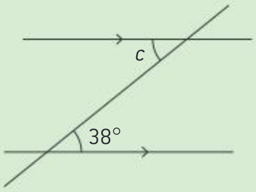

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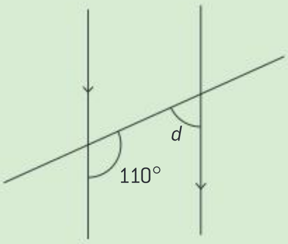
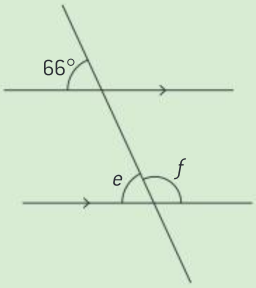


v



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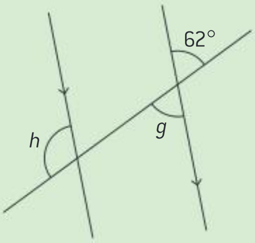
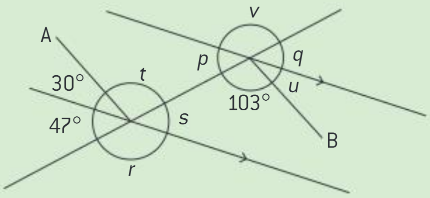


3 Find all angles marked with letters.



Determine whether the lines marked A and B are parallel. Justify your answer.

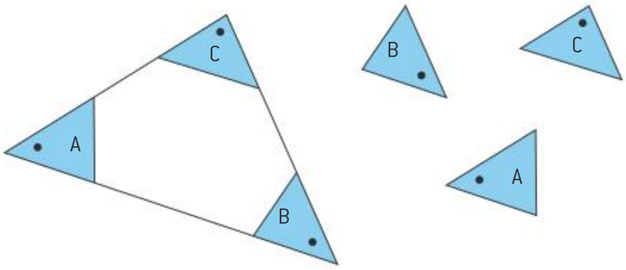
5.2 The triangle sum theorem

Investigation 5.2

a Draw a triangle on a sheet of paper and carefully cut the triangle out with a pair of scissors.

Mark the angles of the triangle A, B and C. (Mark them on both sides). Put a dot in the corner.

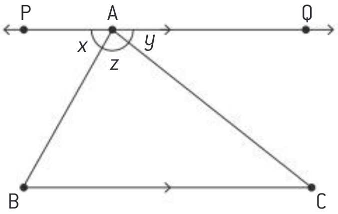
Tear (or cut) the angles from the triangle.



Carefully place the three angles (the points marked with dots) along the side of a ruler. What do you notice? What does this suggest about the three angles of the triangle?

Part **a** was a demonstration of an important result in geometry about the angle sum of a triangle. You have not, however, shown that the result is true for *any* triangle. In part **b**, you will **prove** the result for *all* triangles.

b ABC is a triangle. The line through P and Q passes through A and is parallel to [BC].



i Let angle $P\hat{A}B = x$. Label another angle that is the same size as x. Give a reason.

ii Let angle $Q\hat{A}C = y$. Label another angle that is the same size as y. Give a reason.

iii What is the sum of x, y and z?

iv What does your answer to part **iii** tell you about the angles of a triangle.

Command term

Prove means you should use a sequence of logical steps to obtain the required result in a formal way.

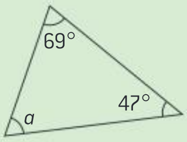
DP ready Applications and Interpretation

If you prefer the approach in part a then you may prefer MAI, but if you prefer the more formal proof in part b then you may prefer MAA.

Exercise 5.2

Find the lettered angles.

1



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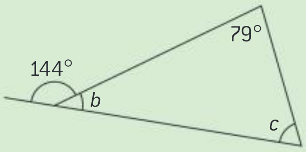
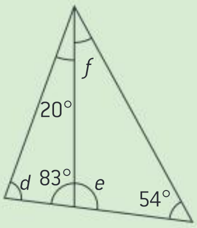


Key point

The angle sum of a triangle is 180°.

3



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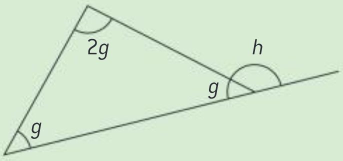

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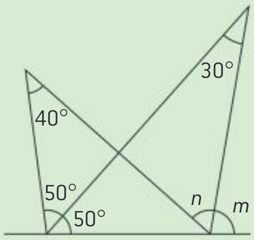

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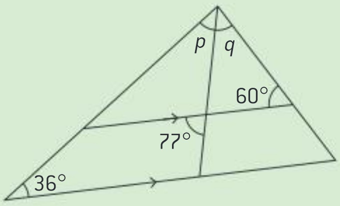
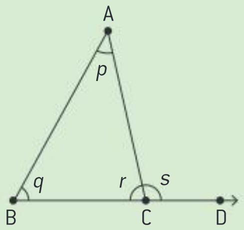


7 ABC is a triangle. Extend the side [BC] to pass through point D.



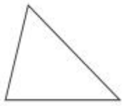
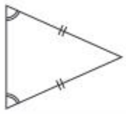
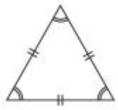
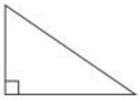
Let $B\hat{A}C = p$ and $A\hat{B}C = q$. Let $A\hat{C}B = r$. Write down a relation between p, q and r.

Let $A\hat{C}D = s$. Write down a relation between r and s.

Hence, find an expression for s in terms of p and q.

5.3 Properties of triangles and quadrilaterals

Types of triangle

Scalene	Isosceles	Equilateral	Right-angled
			
All sides and angles different	Two equal sides and two equal angles at the bases of the equal sides	Three equal sides and three equal angles	Right-angled triangle

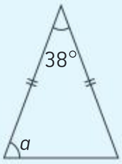
Equal length sides are shown by having the same number of dashes:

Equal size angles are shown by having the same number of arcs:

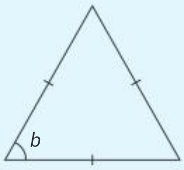
Example 2

Find the lettered angles.

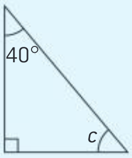
a



b



c

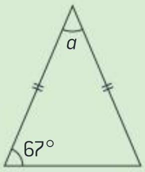


a $a + a + 38 = 180$ $2a = 180 - 38$ $2a = 142$ $a = 71^\circ$	The base angles of the isosceles triangle are equal.
b $b + b + b = 180$ $3b = 180$ $b = 60^\circ$	The three angles of the equilateral triangle are equal.
c $c + 40 + 90 = 180$ $c = 180 - 90 - 40$ $c = 50^\circ$	Note that $40^\circ + 50^\circ = 90^\circ$ Angles of a right-angled triangle are **complementary** (they add up to 90°).

Exercise 5.3a

1 Find the lettered angles.

i



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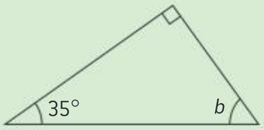

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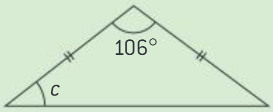
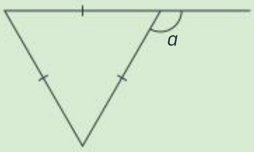
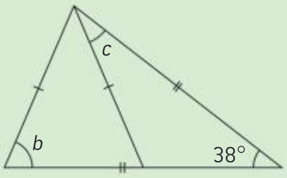


2 Find the lettered angles.

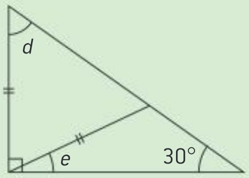
i



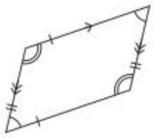
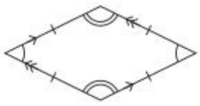
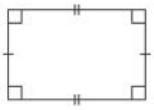
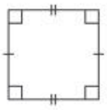
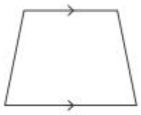
ii



iii



Types of quadrilateral

Parallelogram	Rhombus	Rectangle	Square	Kite	Trapezoid
					
Opposite sides are parallel and the same length opposite angles are equal	All sides are equal and opposite sides are parallel; opposite angles are equal	Opposite sides equal and all angles are 90°	All sides are equal and all angles are 90°	Two pairs of adjacent sides are equal; one pair of opposite angles is equal	Two parallel sides

Investigation 5.3

Draw a parallelogram, a rhombus, a rectangle, a square, a kite and a trapezoid. Your drawings need to be accurate so that sides that should be equal in length are equal, sides that should be parallel are parallel and angles that should be right-angles are right-angles.

(Use an *isosceles* trapezoid, in which the two non-parallel sides are equal in length).

Draw the diagonals of each of the shapes.

Measure the angle between the diagonals.

Measure the lengths of the diagonals.

Each of the diagonals is split into two parts. Measure the lengths of the parts. If a line is split into two equal parts, it is said to be **bisected**.

Use the information you have found to complete this table showing where the diagonals are equal, perpendicular or where they bisect each other.

	parallelogram	rhombus	rectangle	square	kite	trapezoid
equal						
perpendicular						
bisected						

Note

You can draw the shapes using a pencil and ruler, or you can use some geometry software such as Geogebra or Desmos to draw them.

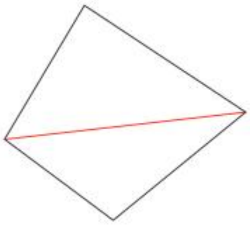
Key point

The sum of the angles of a quadrilateral is 360°.

You saw in the section on triangles that the sum of the angles in any triangle is 180°.

Draw a quadrilateral and draw a diagonal.

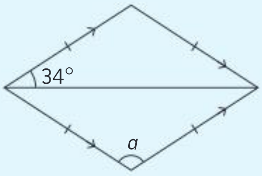
The diagonal divides the quadrilateral into two triangles. The sum of the angles of the quadrilateral will be the sum of the angles of the two triangles, which is $2 \times 180 = 360°$.



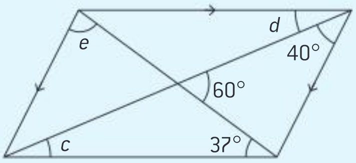
Example 3

Find the lettered angles.

a



b

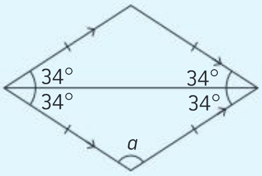


a $2 \times 34 = 68°$

$a = 180 - 68$

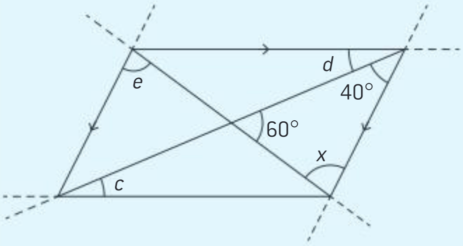
$a = 112°$

Using base angles of an isosceles triangle and alternate angles, the angles shown are all equal.



b $c + 37 + (180 - 60) = 180$

$c = 23°$



$d = 23°$

$180 - (60 + 40) = x$

$x = 80°$

$e = 80°$

c and d are alternate angles.

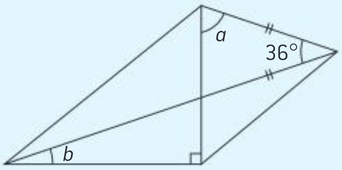
Triangle sum.

e and x are alternate angles.

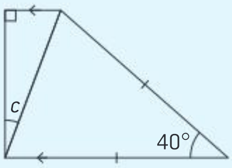
Example 4

Find the angles marked with letters

a



b



a $a = (180 - 36) \div 2$ $= 72°$	Equal angles in an isosceles triangle.
$b = 180 - (90 + 72)$ $= 18°$	Vertically opposite angles and the triangle sum.
b $\frac{1}{2}(180 - 40) = 70$ $90 - 70 = 20°$	Equal angles in an isosceles triangle. Sum of angles in a triangle. Complementary angles.

Exercise 5.3b

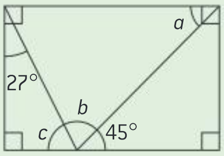
1 Match each of these properties to one or more of these shapes.

i parallelogram, **ii** rhombus, **iii** rectangle, **iv** square, **v** kite, **vi** trapezoid

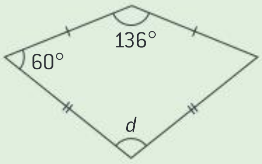
- **a** all angles are equal
- **b** all sides are of equal length
- **c** opposite sides are parallel
- **d** the diagonals bisect each other at 90°
- **e** the diagonals are equal in length
- **f** all angles are 90°
- **g** only one pair of opposite sides is parallel
- **h** two pairs of sides are of equal length
- **i** the diagonals cross at 90°
- **j** only one diagonal bisects the other
- **k** only one pair of diagonally opposite angles is equal
- **l** diagonally opposite angles are equal
- **m** opposite sides are of equal length
- **n** the diagonals bisect each other.

2 Find the lettered angles.

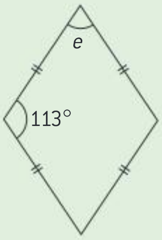
a



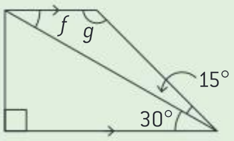
b



c

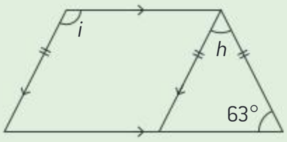


d

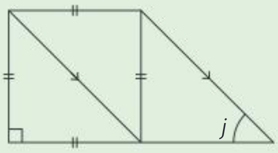


3 Find the lettered angles.

a

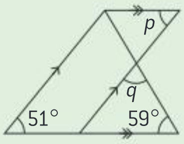


b

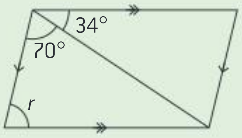


4 Find the lettered angles.

a

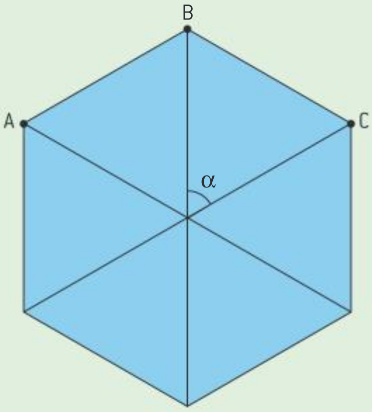


b



DP style **Analysis and Approaches HL**

5 A regular hexagon is made up of 6 triangles as shown.



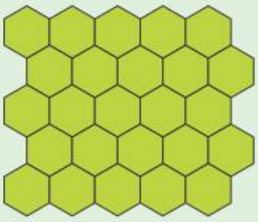
a Find the angle α.

Angle AB̂C is an interior angle of the hexagon.

b Find the size of AB̂C.

An artist wishes to create a pattern using regular polygons that completely covers a flat wall.

c **i** Use your answer to part **b** to explain why it is possible for the artist to use hexagons to cover the wall as shown.

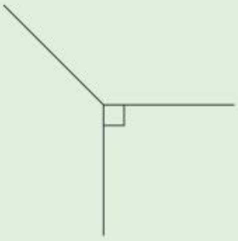


ii State a condition on the interior angle for a regular polygon to completely cover the wall.

iii Show that there are exactly two regular polygons with fewer sides than the hexagon which could completely cover the wall.

iv Explain why there cannot be any other regular polygons that could completely cover the wall.

The artist decides to use two different regular polygons in his pattern. A point where the polygons meet is shown below.

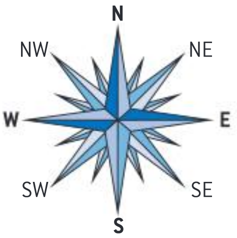


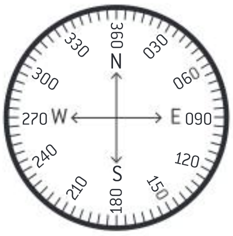
d Given one of the angles is 90˚, find the number of sides in each polygon.

5.4 Compass directions and bearings

The primary compass directions are north, south, east and west and these are subdivided into NE, NW, SE and SW.

This system of subdivisions provides only 8 individual directions and so, for navigational purposes, is not very accurate. A more precise system divides the compass into 360 degrees.





Key point

A **bearing** is an angle measured clockwise from north and written with three digits or figures.

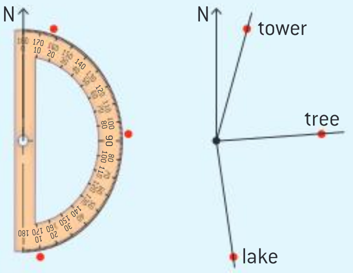
DP ready **International-mindedness**

Early navigators, such as the Polynesian Islanders who, in ancient history, reached Australia, New Zealand and South America, relied on observations of the stars, ocean currents, wind patterns and birds. The magnetic compass was invented in about 200 BCE by the Chinese during the Han dynasty. In later years it was used for navigation.

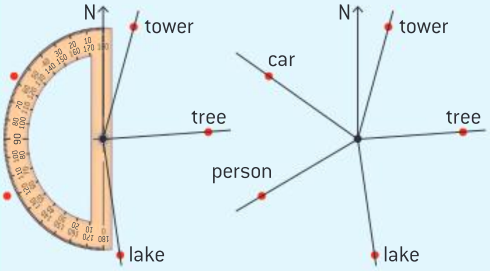
Example 5

From the top of a hill, Heidi observes a tower on a bearing of 017°, a tree on a bearing of 087°, a lake on a bearing of 171°, a person on a bearing of 238° and a car on a bearing of 302°.

Show these directions on a diagram.



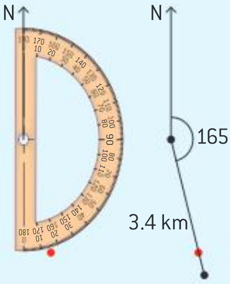
Draw a north line on the page. Measure the angles 17°, 87° and 171° from the north line and draw lines in these directions.



To add the remaining points with a 180° protractor, measure 238 – 180 = 58° and 302 – 180 = 122° and draw lines.

Example 6

A ferry sails a distance of 3.4 km on a bearing of 165°. Draw a diagram to show the ship's journey.



Draw a diagram showing the direction of North.

Draw an angle of 165°.

Measure a scaled distance of 3.4 cm to represent 3.4 km.

DP ready **Theory of knowledge**

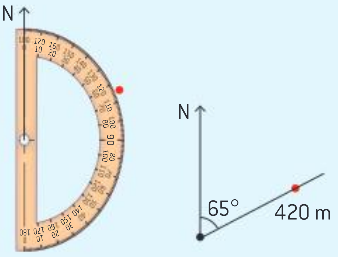
Bearings are used in navigation. Problems are set assuming that they are based on plane geometry. In reality they should be set so that they are on the curved surface of the earth. Consider an aeroplane that travels due north to the North Pole. It then turns through an angle of 90° and flies the same distance due south. To return home it travels due west.

Now look at the diagram. The plane has flown along three sides of a triangle. Each of the angles of the triangle is 90°, so their sum is 270°, not 180°.

On a small scale, and not close to the poles, it is acceptable to approximate journeys on the earth's surface, as long as it is not necessary to take the curvature into account. What we assume, that the angles of a triangle will always add up to 180°, is not necessarily always true.

Example 7

Arjun walks 420 m on a bearing of 065°. He turns to a direction of 205° and walks a further 330 m. Draw a diagram to show Arjun's journey.

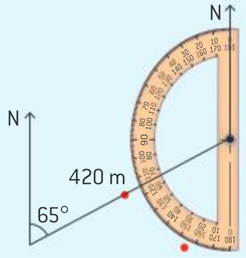


Draw a diagram showing the direction of North.

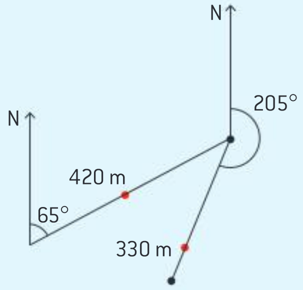
Draw an angle of 65°.

Measure a scaled distance of 4.2 cm to represent 420 m.

Draw another north line (parallel to the first).



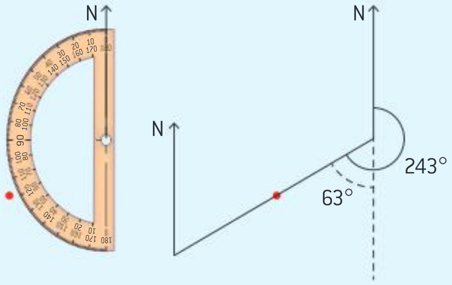
Draw an angle of 205°. (To do this measure an angle of 205 – 180 = 25°)



Measure a scaled distance of 3.3 cm to represent 330 m.

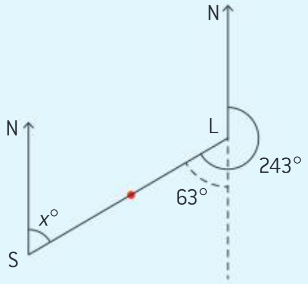
Example 8

An observer in a lighthouse sees a ship in the distance on a bearing of 243°. What is the bearing of the lighthouse from the ship?



Draw a diagram, showing direction of North. Draw an angle of 243° to show the bearing of the ship.

To draw the angle of 243°, measure 243 – 180 = 63°.



The bearing of the lighthouse from the ship is x°.

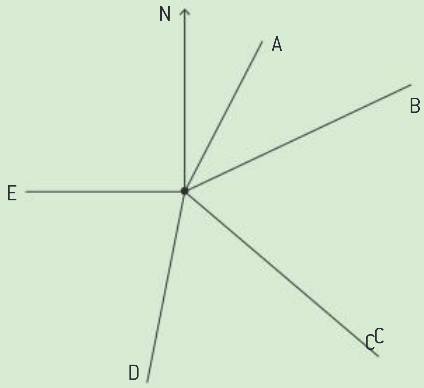
Since the north lines are parallel, then, by alternate angles

$x = 63$.

The bearing is 063°.

Exercise 5.4

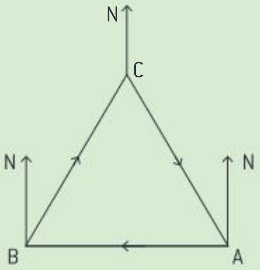
1 Match the letters A, B, C, D, E to these bearings.



270°, 192°, 027°, 131°, 064°.

2 A ship sails 37 km on a bearing of 008° and then changes direction to sail 45 km on a bearing of 079°. Using a scale of 1cm to 1 km, draw a diagram to show the ship's journey.

3 A plane flies round three sides of an equilateral triangle.



The journey starts at A and the first section of the journey is due West.

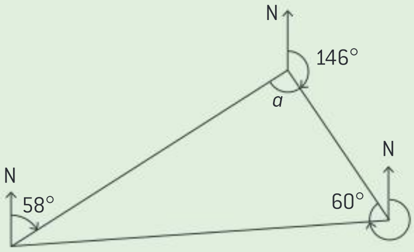
Find the bearings of the three sections of the journey.

4 A man cycles 12 km on a bearing of 137°. On what bearing must he travel to return the same way?

5 The bearing of a plane from the control tower is 327°. In which direction must the plane travel towards the control tower?

DP style **Applications and Interpretation SL**

6



A triangular circuit is marked out on a flat piece of ground. The first section of the circuit is on a bearing of 058° and the second is 146°.

i Calculate the angle, *a* between the first and second section of the circuit.

The angle between the second and third section of the circuit is 60°

ii Calculate the bearing, *b* of the final section.

5.5 Geometric transformations

Classical Euclidean geometry was dominant for about 2000 years. Mathematicians began to develop other ways of looking at geometry much later. Transformation geometry was developed in the nineteenth and twentieth centuries. Transformations include **reflections**, **rotations**, **translations** and **enlargement**.

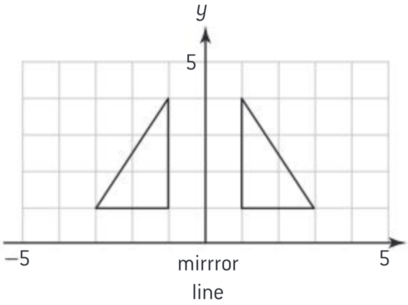
Reflection

If you look in a mirror, you see a reflection.

Key point

In a reflection, points in the image are the same perpendicular distance from the mirror line as the corresponding points in the object. The object and the image are congruent.

A mathematical reflection also produces an image, the same distance behind the mirror in the same way.

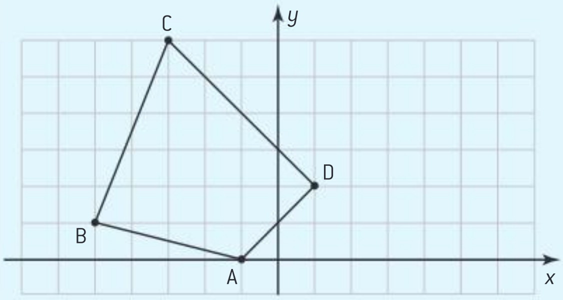


Key point

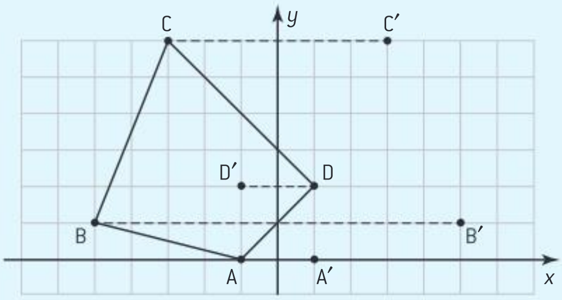
Two shapes are congruent if they are the same size and shape.

Congruent shapes fit exactly on to one another, though you may need to rotate or flip them. These two hand shapes are congruent.

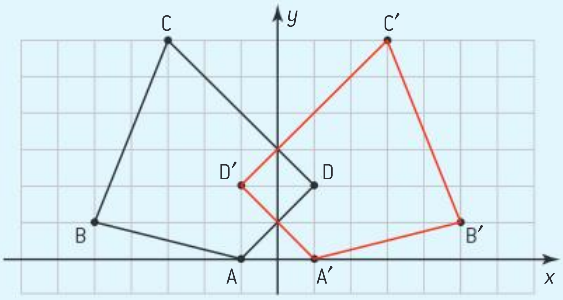
Example 9



Reflect the quadrilateral ABCD in the y-axis.

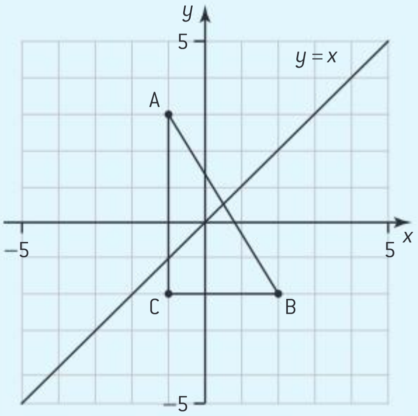


Construct points A′, B′, C′, D′ on the opposite side of the y-axis so that they are the same perpendicular distance from the line.

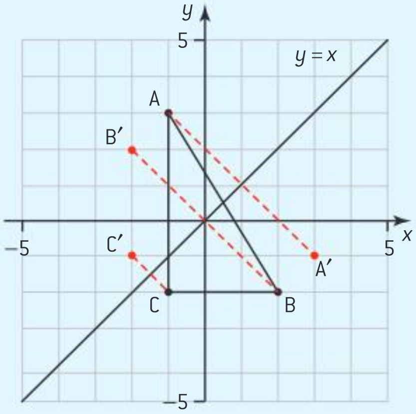


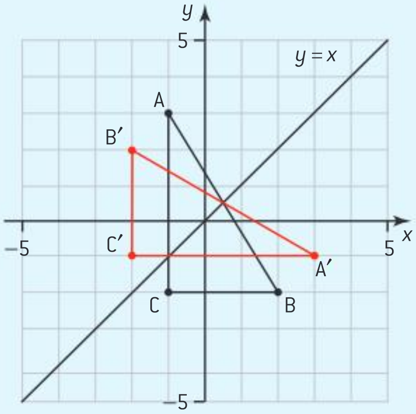
Join the points to form the reflected image A′B′C′D′.

Example 10



Reflect the triangle ABC in the line $y = x$.





Construct points A′, B′, C′ on the opposite side of the line $y = x$ so that they are the same perpendicular distance from the line.

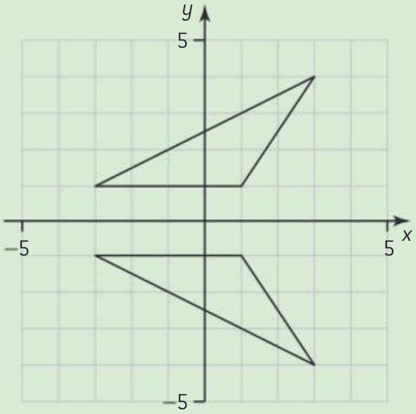
You can measure this with the diagonal distances across the squares.

Join the points to form the reflected image A′B′C′D′.

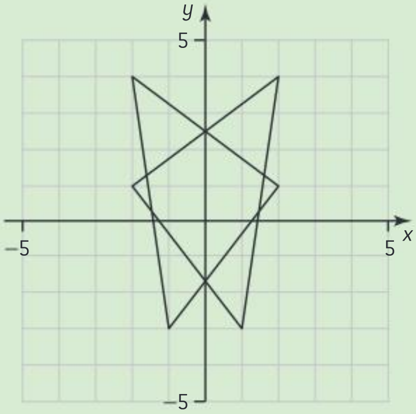
Exercise 5.5a

1 Write down the equation of the mirror line.

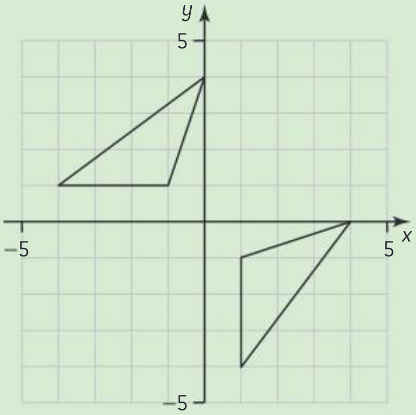
a



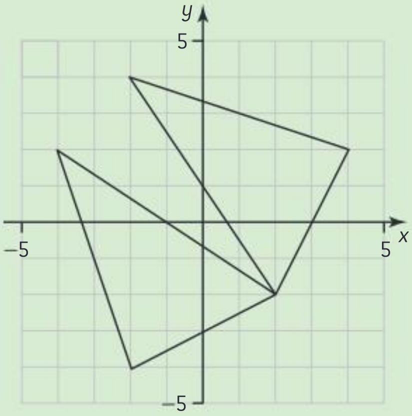
b



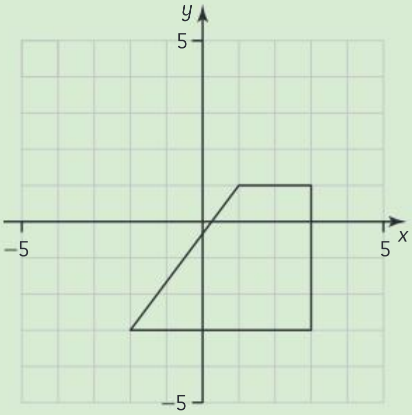
c

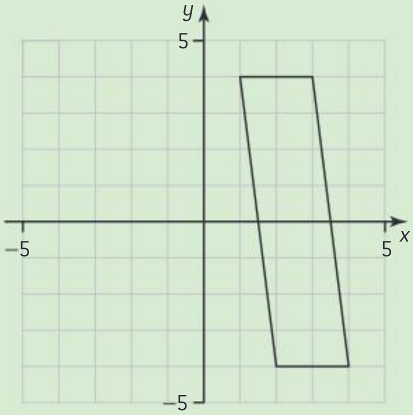


d

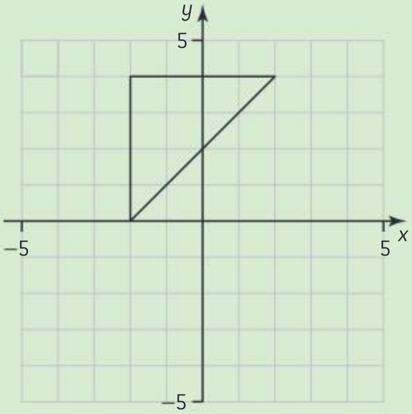


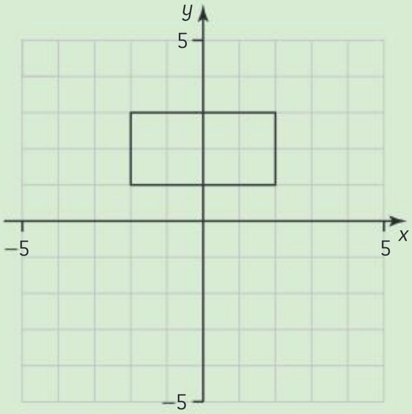
2 **a** Reflect this shape in $y = 0$. **b** Reflect this shape in $x = 0$.





c Reflect this shape in $y = x$. **d** Reflect this shape in $y = -x$.





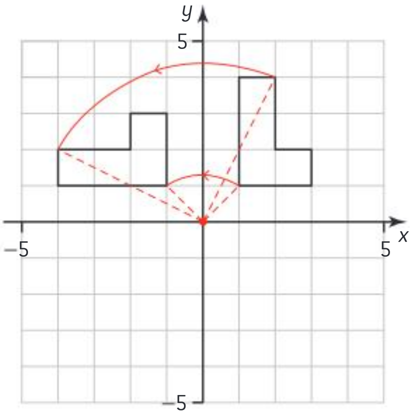
Key point

A rotation is a circular motion around a central point: the centre of rotation. As with reflections, the object and the image are again congruent.

Note

Unlike with reflections, the image of an object after rotation is directly congruent: you don't have to flip the image for it to fit onto the object.

Rotations

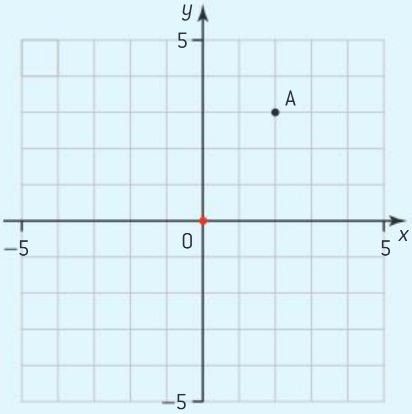


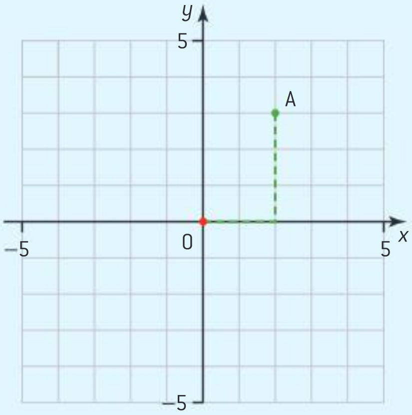
In order to describe a rotation, you need to state where the centre of rotation is, the angle of rotation and its direction (clockwise or anti-clockwise).

An anticlockwise rotation is a positive rotation and a clockwise rotation is a negative rotation. For example, a clockwise rotation of 90° is – 90° and an anticlockwise 90° rotation is 90°.

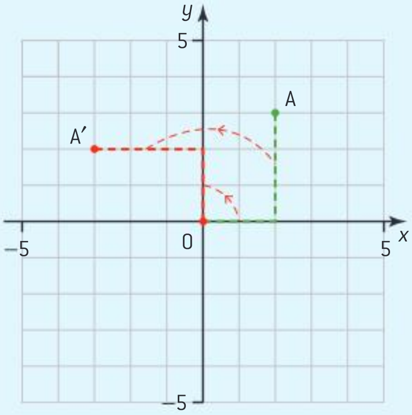
Example 11

Rotate the point A 90° anticlockwise (+90°) and 180°, centred at (0, 0).

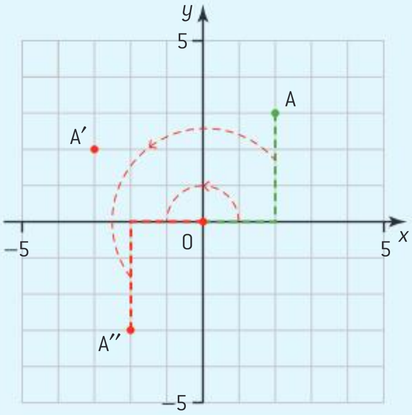




Draw an ‘L’ shape between (0, 0) and the point A. The two parts of the ‘L’ are in the *x*- and *y*-directions.



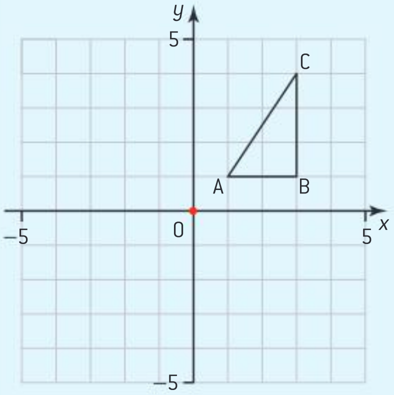
Rotate the ‘L’ shape through 90°, centred at (0, 0).

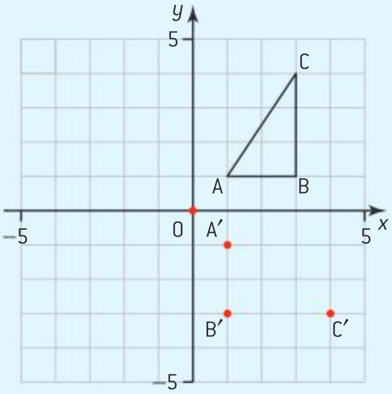


Rotate the ‘L’ shape through 180°, centred at (0, 0).

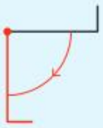
Example 12

Rotate the triangle ABC 90° clockwise (– 90°) centred at (0, 0).

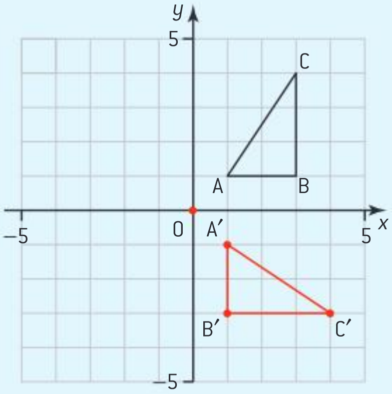




Either use the 'L' shape for each of the points A, B or C or, trace the triangle ABC on some tracing paper and, holding the centre of rotation in position with the point of a pair of compasses, rotate the shape through – 90°.



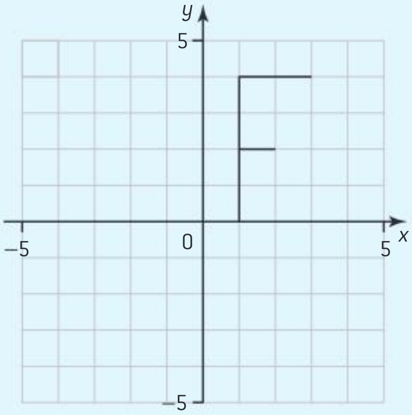
Mark the three points A′, B′ and C′.

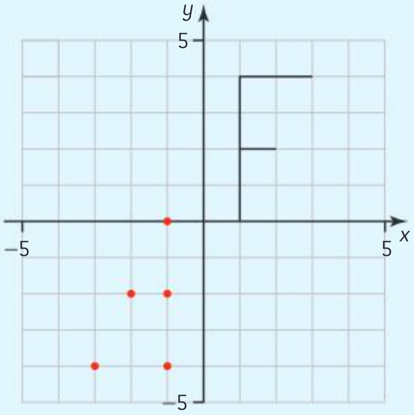


Join the points A′, B′, C′ to form the image.

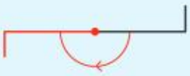
Example 13

Rotate the shape through 180° about (0, 0).

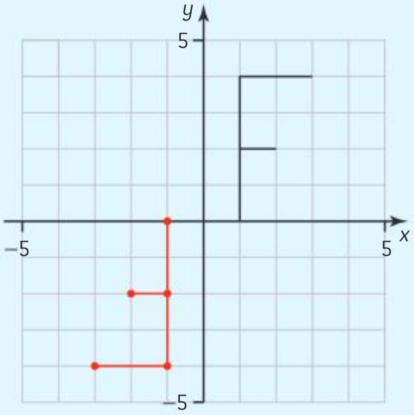




Either use the ‘L’ shape for each of the points or, trace the shape on some tracing paper and, holding the centre of rotation in position with the point of a pair of compasses, rotate the shape through 180°.



Mark the points.



Join the points to form the image.

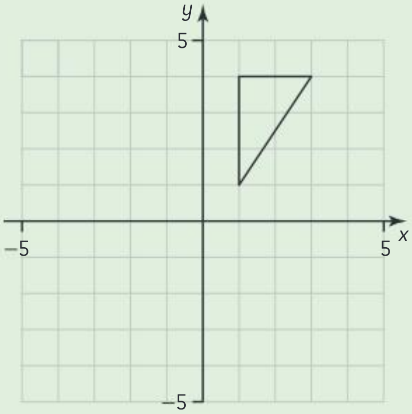
Exercise 5.5b

1 **a** Rotate the point (2, 3) 90° clockwise around (0, 0).

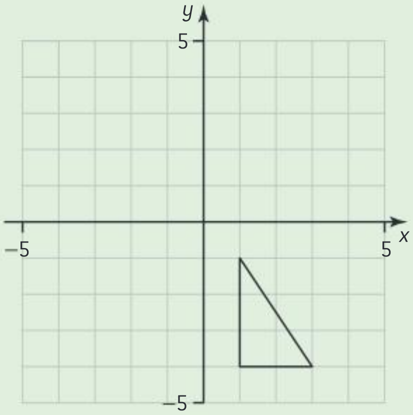
  b Rotate the point (–3, –2) 90° anticlockwise around (0, 0).

  c Rotate the point (–4, 1) 180° around (0, 0).

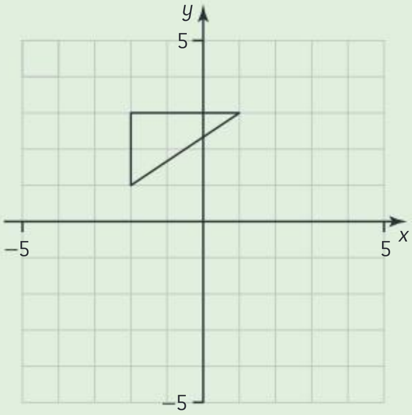
2 **a** Rotate the shape 90° anticlockwise around (0, 0).



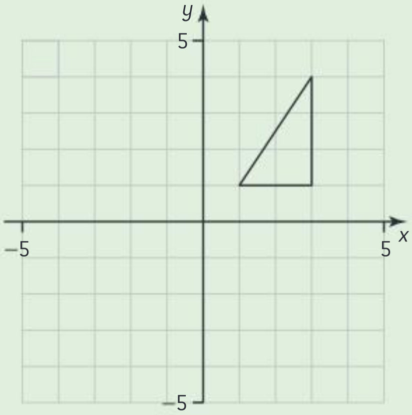
  b Rotate the shape 90° clockwise around (0, 0).



c Rotate the shape by 180° around (0, 0).



3 Reflect this shape in $y = 0$ and then reflect the image in $x = 0$.



Describe the single transformation that would take the original shape to the second image.

DP style **Applications and Interpretation HL**

4 **a** Write down the image of the point (a, b) when it undergoes the following transformations

- **i** Reflection in the y-axis
- **ii** Reflection in the line $y = x$
- **iii** Reflection in the line $y = -x$

Finding the image of (a, b) under a reflection in the y-axis can be described in words as 'changing the sign of the x coordinate and keeping the y-coordinate the same'.

b Use this definition to demonstrate that reflecting twice in the y-axis returns the point to (a, b).

c Describe the effects on the x and y coordinates of a reflection in the line

i $y = x$ **ii** $y = -x$

d **Hence** give the coordinates of (a, b) after the point has been reflected twice, first in the line $y = x$ and then in the line $y = -x$

e Use the L method to find the coordinates of (a, b) after a rotation of 90° anticlockwise about (0, 0) and describe the effect on the x and y coordinates of this transformation.

f **Hence** give the coordinates of the image of (a, b) after

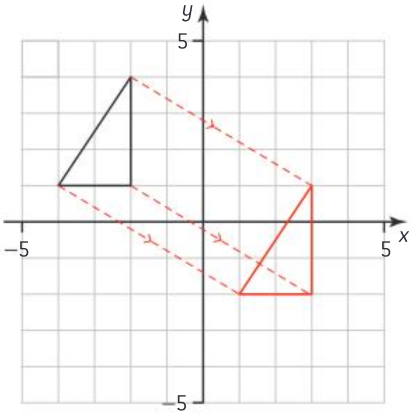
- **i** two rotations of 90°
- **ii** three rotations of 90°
- **iii** four rotations of 90°

g **i** Explain the result in part **f iii**

ii Use your answer to part **f** to give a single transformation equivalent to a reflection in the line $y = x$ followed by a reflection in the line $y = -x$

Translations

A third transformation that produces a congruent image is a translation.



The triangle in the diagram has been shifted 5 units to the right and 3 units down.

Key point

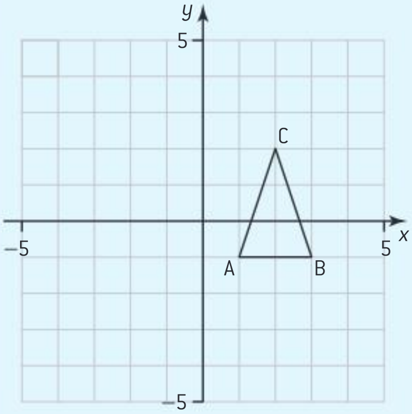
A translation is a 'sliding' transformation. The object slides on the grid without rotating or reflecting it.

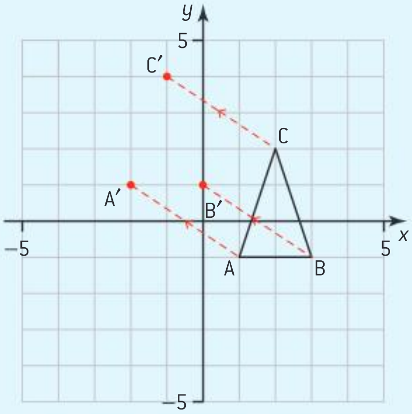
DP link

Students studying the DP will learn more about vectors at HL

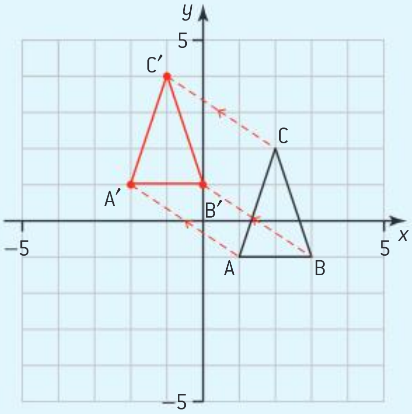
Example 14

Show the translation of this triangle 3 units to the left and 2 up.





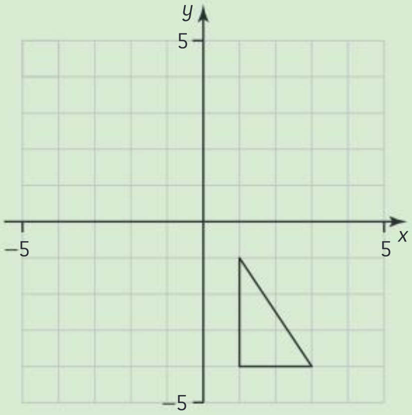
Translate each of the points A, B, C.



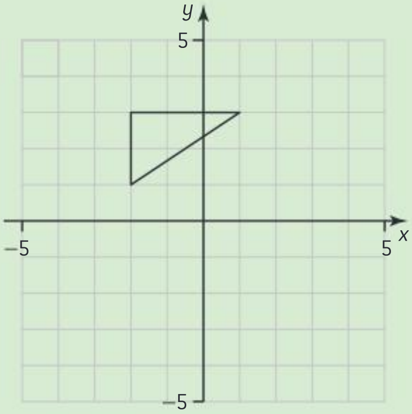
Join the points A′, B′, C′ to form the image.

Exercise 5.5c

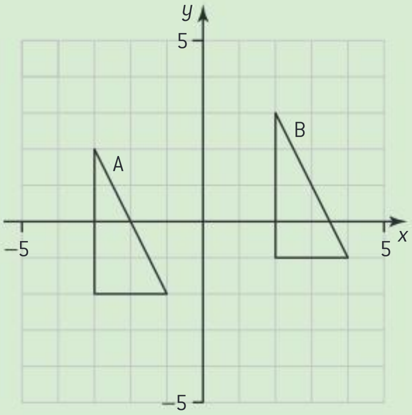
1 Translate this shape 2 units to the left and 4 units up.



2 Translate this shape 3 units to the right and 4 units down.



3 Describe the translation that transforms A to B.



4 State whether the image of an object transformed by each of the following is directly or oppositely congruent.

a Reflection

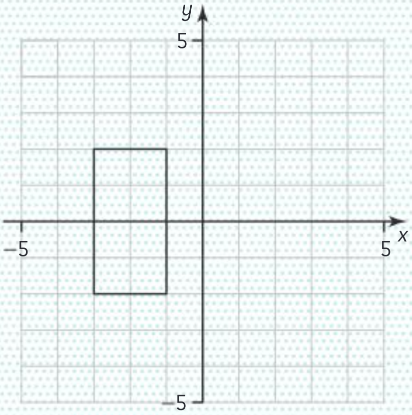
b Rotation

c Translation

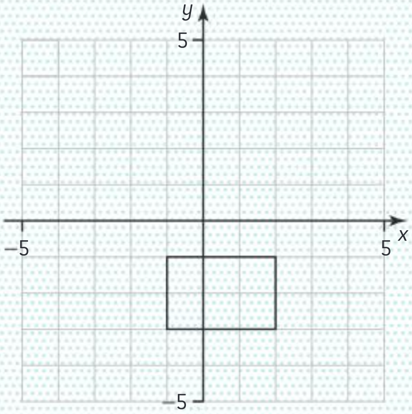
Higher Level

5 A translation x units to the right and y units up can be written using a **vector** $\begin{pmatrix} x \\ y \end{pmatrix}$. Negative values of x or y shift to the left or down.

a Translate the shape by the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Copy and complete the diagram, showing the image after translation.

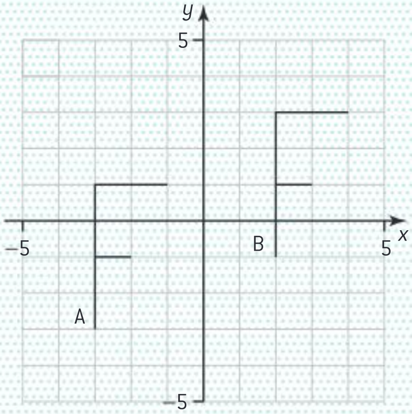


b Translate the shape by the vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Copy and complete the diagram, showing the image after translation.

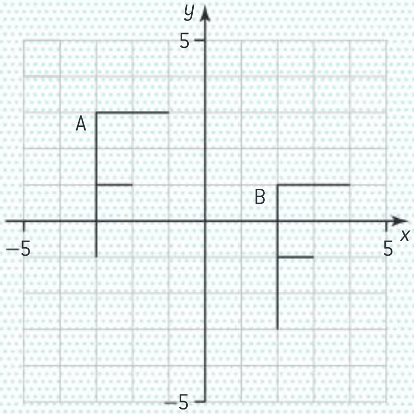


6 Describe these transformations from A to B using vectors.

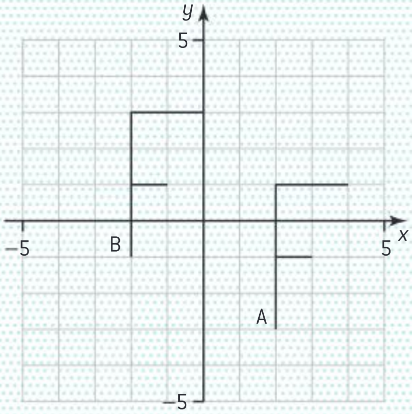
a



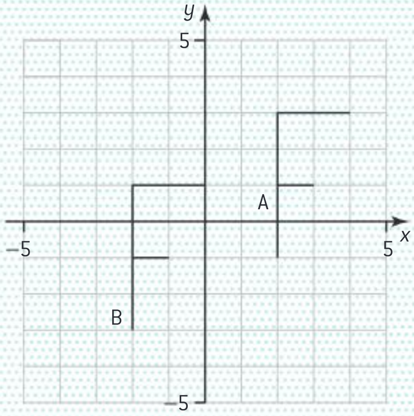
b



c



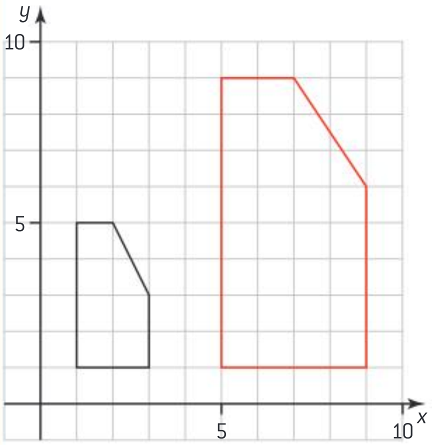
d

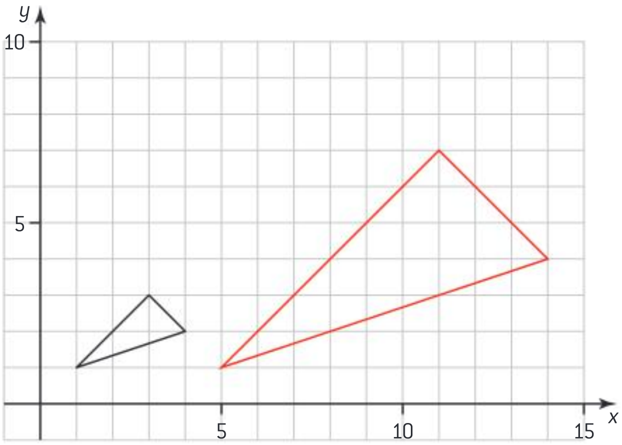


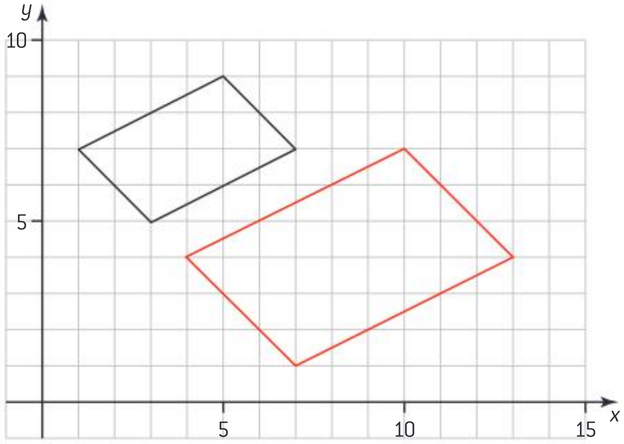
Enlargements

Investigation 5.4

These three diagrams show a shape and its image after an enlargement and translation.







Key point

Two shapes are similar if corresponding lengths are in the same ratio. Corresponding angles will be the same. The shape will be the same, but a different size.

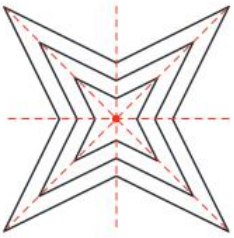
a Describe the shapes and their images.

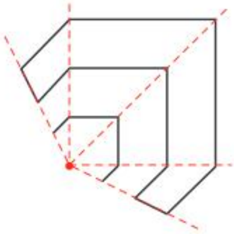
b Measure the lengths of the sides of the original shapes and their images.
Find the ratios of each pair of lengths.
What do you notice?

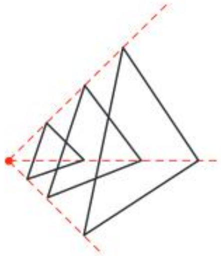
c Measure the angles of the original shapes and their images.
What do you notice?

Are the objects and the images *similar*?

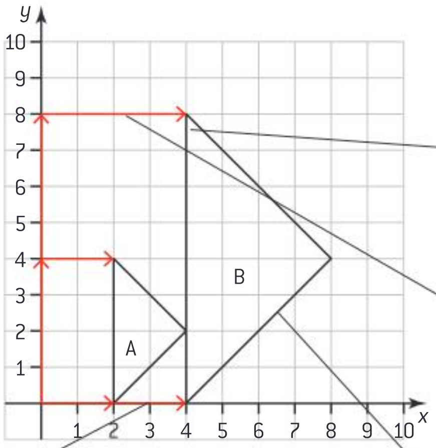
When you enlarge a shape, the ratio of the lengths of the sides is the scale factor of the enlargement. The enlarged shape is similar to the original shape. There is also a point in any enlargement that is a fixed point. Imagine all the points of the shape expanding outwards from this point which is called the centre of enlargement. The centre may be in the middle of the object, at a corner or it may be outside the object.







You can enlarge a shape drawn on a set of axes from the centre of enlargement. The diagram below shows an enlargement with scale factor 2 and centre (0,0).



Count the squares from the centre of enlargement to each vertex. Multiply all the distances from the centre of enlargement by the scale factor.

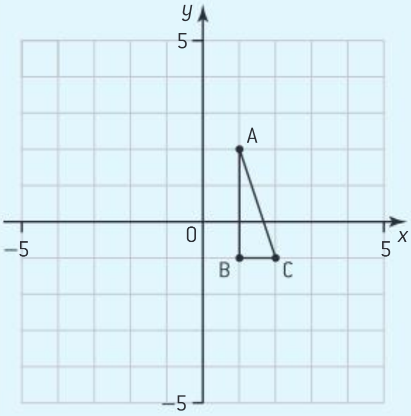
The distance to the top vertex changes from 4 up 2 right to 8 up, 4 right.

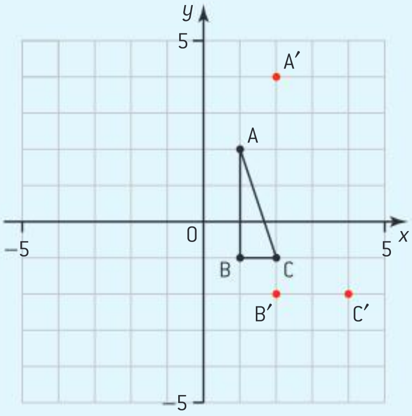
The distance to the bottom vertex changes from 2 right to 4 right.

Check that the lengths of the image are twice as long as the original.

Example 15

Enlarge this triangle ABC by a scale factor of 2 around the point (0, 0), where A is (1, 2), B is (1, –1) and C is (2, –1).

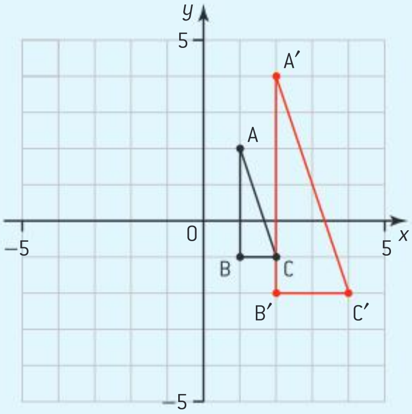




Find the distance from the centre of enlargement (0, 0) to each of the vertices of the shape.

Since the centre of enlargement is the origin, the coordinates themselves tell you the distance.

You can multiply the coordinates by the scale factor.

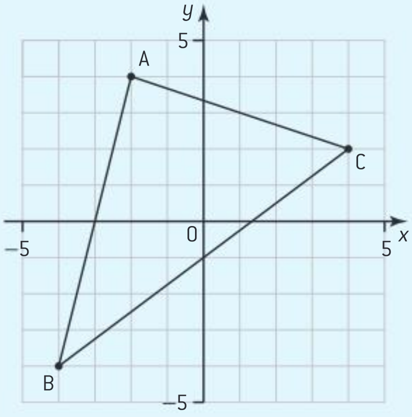


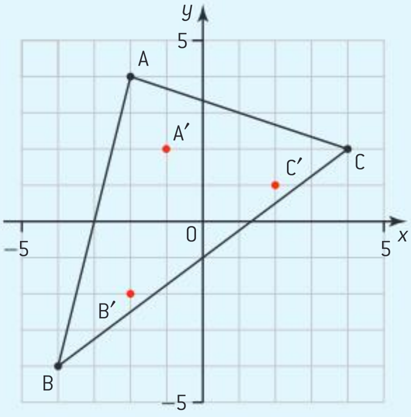
A′ is (2, 4), B′ is (2, –2) and C′ is (4, –2).

Join the points A′, B′, C′ to form the image.

Example 16

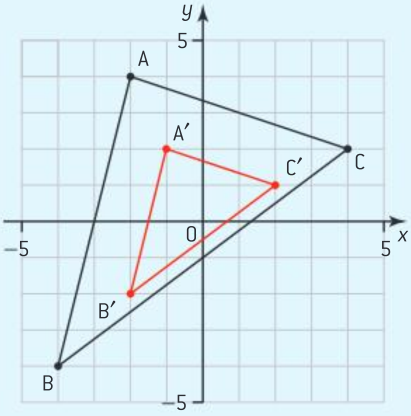
Enlarge this triangle ABC by a scale factor of $\frac{1}{2}$ around the point (0, 0), where A is (–2, 4), B is (–4, –4) and C is (4, 2).





Again, centre of enlargement is (0, 0) so multiply the coordinates by the scale factor.

A′ is (–1, 2), B′ is (–2, –2) and C′ is (2, 1).



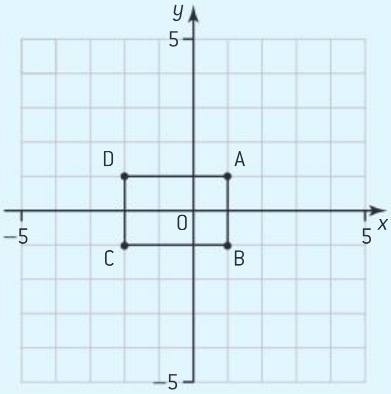
Join the points A′, B′, C′ to form the image.

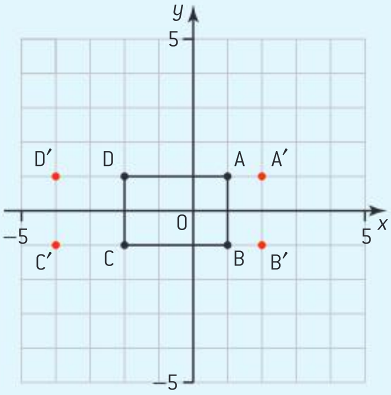
You will notice that as the enlargement has a scale factor that is a fraction, it forms an image that is *smaller* than the original shape.

Another transformation that is like an enlargement is a **stretch**. Unlike an enlargement, the image is not similar to the original shape, but is distorted. A stretch takes place in one direction only and has a fixed line that is perpendicular to the direction of the stretch.

Example 17

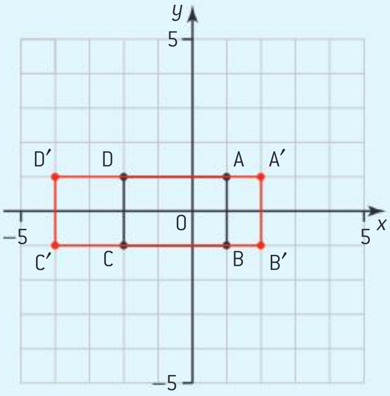
Stretch this rectangle ABCD by a scale factor of 2 in the x-direction, away from the y-axis where A is (1, 1), B is (1, –1), C is (–2, –1) and D is (–2, 1).





Since the centre of enlargement is the line $x = 0$, multiply the x-coordinates by the scale factor.

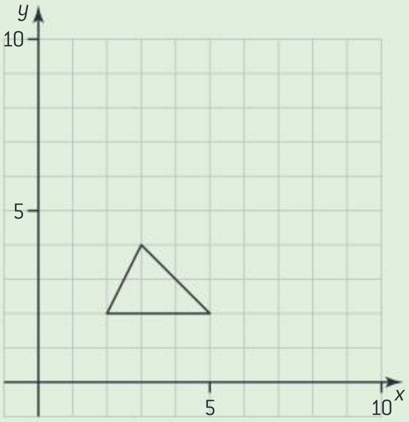
A′ is (2, 1), B′ is (2, –1), C′ is (–4, –1) and D′ is (–4, 1).



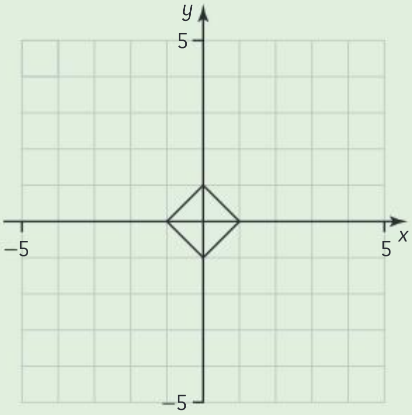
Join the points A′, B′, C′ and D′ to form the image.

Exercise 5.5d

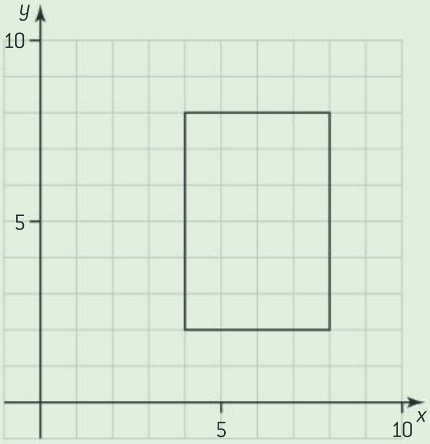
1 Enlarge the shape with a scale factor of 2 about the point (0, 0).



2 Enlarge the shape with a scale factor of 3 about the point (0, 0).

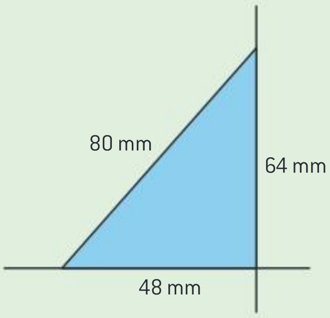


3 Enlarge the shape with a scale factor of $\frac{1}{2}$ about the point (0, 0).



DP style **Applications and Interpretation HL**

4 The design for a logo consist of a right-angled triangle with sides of lengths 48 mm, 64 mm and 80 mm as shown in the diagram below.



This triangle is rotated 90° clockwise about the vertex at the right-angle and enlarged by a scale factor of $\frac{3}{4}$.

a Show the two triangles on a single diagram.

b Write down the lengths of the sides of the new triangle.

The same transformation is then performed on the new triangle, and then again on its image so that the full logo consists of 4 triangles of decreasing size.

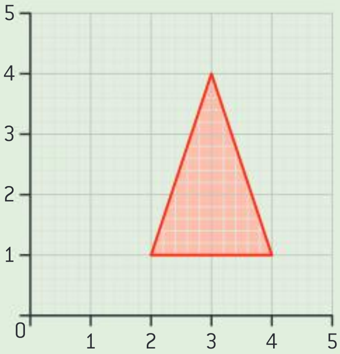
c Add the final two triangles to the diagram drawn in part **a**.

d Find the lengths of the sides of these two triangles.

e Find the perimeter of the logo.

DP style **Analysis and Approaches HL**

5 A triangle has vertices at (2, 1), (3, 4) and (4, 1).



a Find the area of the triangle.

The triangle is enlarged by a scale factor of 2, with a centre at (0, 0).

b **i** Sketch the position of the triangle after the enlargement.

ii Find the area of the new triangle.

c A triangle with area A, base b and height h is enlarged by a scale factor p. Prove the area of the enlarged triangle is p^2A

Chapter summary

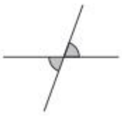
☐ A point is a dimensionless location in space.

A line is one-dimensional and extends infinitely in both directions.

A line segment is a line with two endpoints.

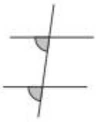
☐ Two lines are parallel if they are the same distance apart and never touch.

☐ When two lines cross:

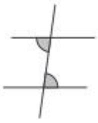


Vertically opposite angles are equal.

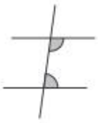
☐ When a line crosses two parallel lines:



Corresponding angles are equal.



Alternate angles are equal.



Co-interior angles are supplementary.

☐ The angle sum of a triangle is 180°.

☐ The sum of the angles of a quadrilateral is 360°.

☐ A **bearing** is an angle measured clockwise from north and written with three digits or figures.

☐ Two shapes are **congruent** if they are the same size and shape.

☐ Two shapes are **similar** if corresponding lengths are in the same ratio. Corresponding angles will be the same. The shape will be the same, but a different size.

☐ In a reflection, points in the image are the same perpendicular distance from the mirror line as the corresponding points in the object. The object and the image are congruent.

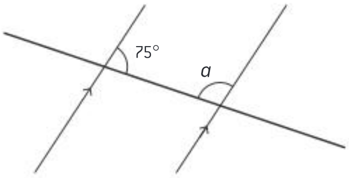
☐ A rotation is a circular motion around a central point: the centre of rotation. The image is again congruent but, unlike the reflection, a rotation is directly congruent.

☐ A translation is a 'sliding' transformation. The object slides on the grid without rotating or reflecting it.

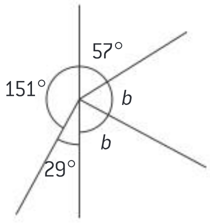
☐ When you enlarge a shape, the ratio of the lengths of the sides is the scale factor of the enlargement. The fixed point from which the enlargement takes place is the centre of enlargement.

Chapter 5 test

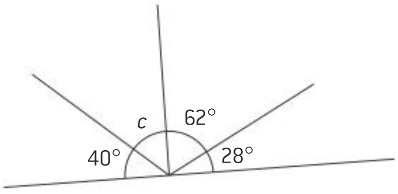
1 Find the angle *a*.



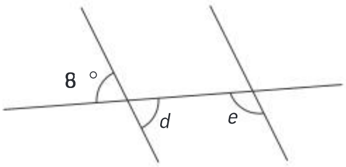
2 Calculate the angle marked *b*.



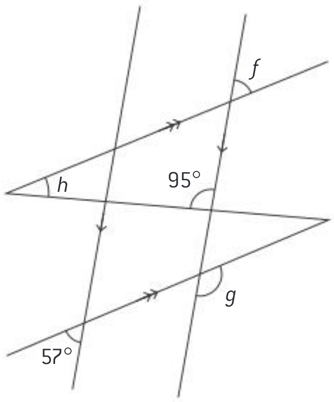
3 Calculate the angle marked *c*.



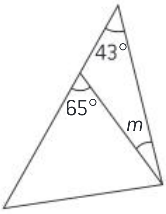
4 Find the lettered angles.



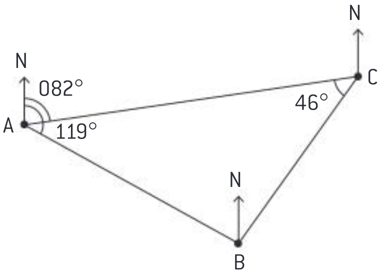
5 Find the lettered angles.



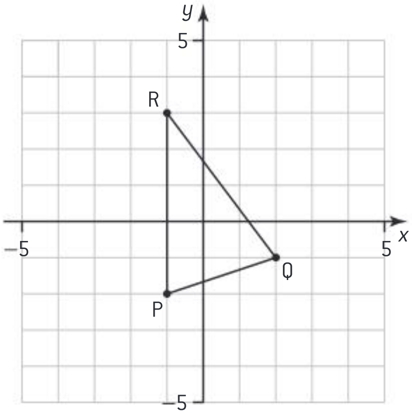
6 Find the angle m.



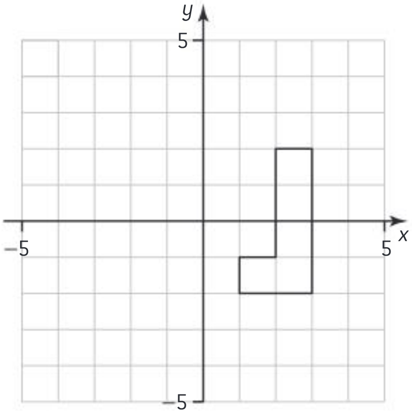
7 From A the bearing of B is 119° and the bearing from C is 082°. The angle between [AC] and [BC] is 46°. Calculate the bearing of C from B.



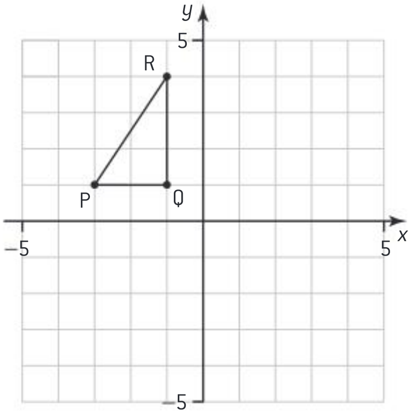
8 Reflect the shape PQR in the line $x = 0$.



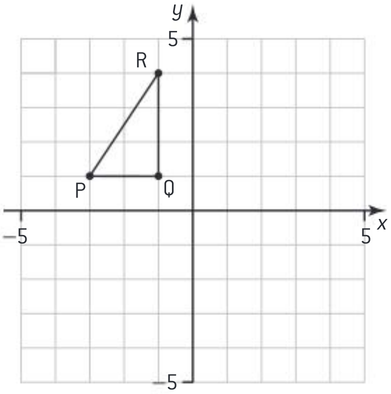
9 Reflect this shape in the line $y = x$.



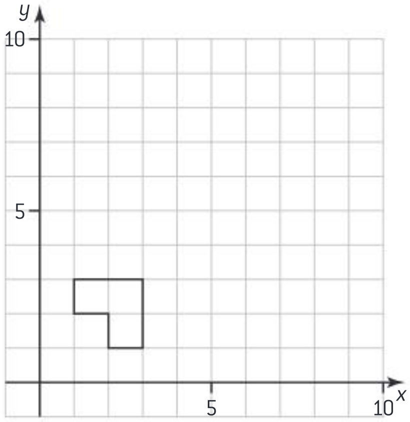
10 Rotate the shape PQR through 90° clockwise about the point (0, 0).



11 Translate the shape PQR by 3 units in the x-direction and –3 units in the y-direction.



12 Enlarge this shape with scale factor 3 about the point (0, 0).



DP style Applications and Interpretation SL

13 The square, A, with vertices at the points (1, 0), (1, 1), (2, 1) and (2, 0) is transformed by a one-way stretch parallel to the x-axis, scale factor 3.

a Plot the position of the square A and its image A′.

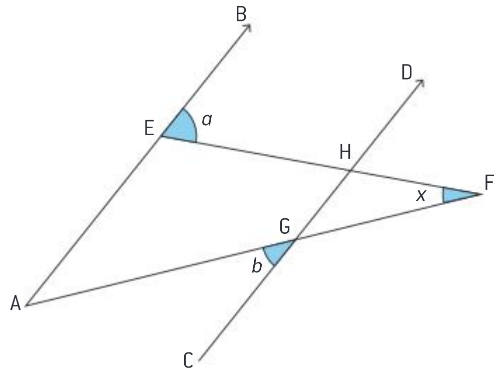
A′ is now translated 3 units to the right. The image of A′ is A″.

b Plot the position of A″.

c Square A is translated d units to the right and then stretched parallel to the x-axis, scale factor 3 to give image B. Find the value of d if B is in the same position as A″.

DP style Analysis and Approaches

14 In the diagram below line [AB] is parallel to line [CD].



a Find the value of x given that $a = 62°$ and $b = 37°$.

b Prove that $x = a - b$.

Modelling and investigation

DP ready Approaches to learning

Critical thinking: Analyzing and evaluating issues and ideas

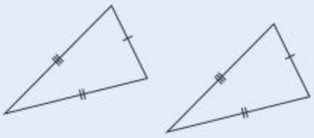
Organization skills: Managing time and tasks effectively

Congruent triangles

If you like this type of problem where you have to prove results, then you may prefer the MAA course.

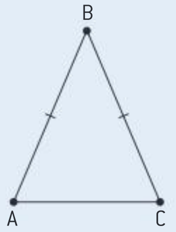
You will recall the definition of congruence, that if two shapes are congruent, then they will have the same size and shape. So, if two triangles are congruent, then they will have equal sides and equal angles. It is not, however, necessary to show that all three sides and all three angles are equal in order to prove congruence.

1 One way to do this is to show that each of the three sides are equal (SSS).



You could prove this to be the case using more basic properties of the triangle, but instead you will use this property to show another property of an isosceles triangle.

To begin, define an isosceles triangle to have two equal sides. You will prove that, if this is the case, then the triangle will also have two equal angles.



ΔABC is a triangle in which AB = CB

Take a point D which is the midpoint of AC.

a What can you say about the lines AD and CD?

b What can you say about the sides of the triangles ΔABD and ΔCBD?

c What does this tell you about the two triangles? (Give a reason)

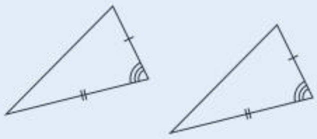
d What does this tell you about the angles $B\hat{A}D$ and $B\hat{C}D$?

e What does this tell you about the angles $B\hat{A}C$ and $B\hat{C}A$?

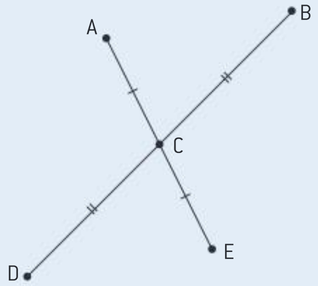
You should have proved that a triangle with equal sides has equal angles.

2 A second way to show congruence is for there to be two equal sides and the included angle equal. (SAS)

(If the angle is not the one that is between the two sides then congruence does not follow).



You will use this property to prove that if two straight lines bisect each other then they form two congruent triangles.



Join AB and ED to form two triangles.

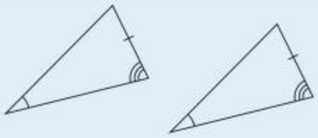
a What can you say about AC and EC?

b What can you say about BC and DC?

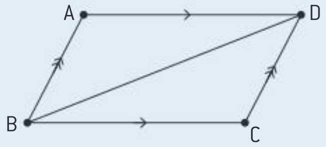
c What can you say about $A\hat{C}B$ and $E\hat{C}D$? Justify your answer.

d What does this tell you about the two triangles? Justify your answer.

3 A third set of conditions for congruence is for two angles and a side equal. (AAS)



You will use this property to prove that a parallelogram has opposite sides that are equal in length.



Add the diagonal BD to the parallelogram.

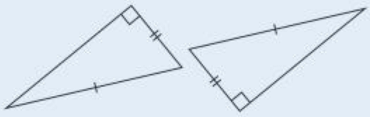
a What can you say about $A\hat{B}D$ and $C\hat{D}B$? Justify your answer.

b What can you say about $A\hat{D}B$ and $C\hat{B}D$? Justify your answer.

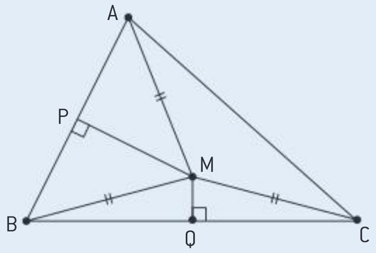
c Since BD is common, what can you say about the triangles ΔADB and ΔCBD? Justify your answer.

d What have you found about the lines AB and CD and the lines AD and BC?

4 If the hypotenuse and one side of two right-angled triangles are equal, then they are congruent. (RHS) (Unlike SAS, the angle does not need to be between the two sides)



You will use this property to find the position of a point that is equidistant from the vertices of a triangle. (The circumcentre of the triangle)



M is a point that is equidistant from the vertices of the triangle ABC. MA = MB = MC. MQ is drawn perpendicular to BC and MP is drawn perpendicular to AB.

a Show that ΔAMP and ΔBMP are congruent using the RHS property.

b Show that ΔBMQ and ΔCMQ are congruent using the RHS property.

c Show that PM and QM are the **perpendicular bisectors** of AB and BC respectively.

d What would happen if you drew a circle with centre M through A?

6

Right-angled triangles

Learning outcomes

In this chapter you will learn about:

- Pythagoras' theorem and its converse
- Mid-point of a line segment and the distance between two points in the Cartesian plane
- Right-angle trigonometry, including simple applications for solving triangles

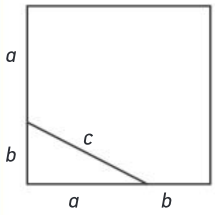
Key terms

- Hypotenuse
- Converse
- Opposite
- Adjacent
- Sine
- Cosine
- Tangent

6.1 Pythagoras' theorem

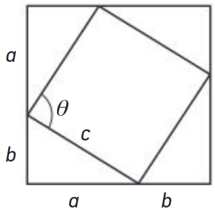
Investigation 6.1

A triangle with sides a, b and c is drawn in one corner of a square.



The length of the side b is equal to the distance cut off from the side of the square by a. The sides of the square are $a + b$.

Three more identical triangles are drawn.



1 Find the angle marked θ in the diagram.

2 Describe the shape that is left if the four triangles were to be cut away.

DP link

This investigation is similar in style to those you will meet if you study the MAA course.

DP ready **Theory of knowledge**

Pythagoras' understanding of number was that all numbers were rational and could be expressed as the ratio of integers. It is believed that the person who showed that the diagonal of a unit square is irrational was Hippasus, one of Pythagoras' followers who lived maybe a hundred years after the death of Pythagoras.

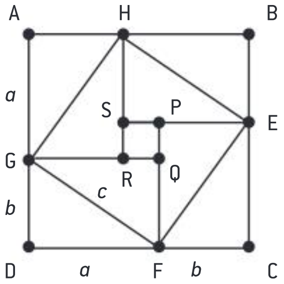
Since this undermined one of the Pythagoreans' basic beliefs of number, Hippasus was, according to the story, drowned at sea.

This is one of the earliest instances of mathematicians or scientists making discoveries which went against the teaching of their times and facing extreme consequences. Axiomatic systems (which you learned about in chapter 5) can be challenged when the 'self-evident truths' they are based on are challenged.

DP link

In the MAA course you will prove that $\sqrt{2}$ is an irrational number.

Four more triangles are drawn, congruent to the first four, as shown.



3 Write down the lengths of GQ and SE.

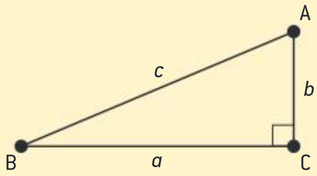
4 Using SE + GQ – SP = DC, form an equation involving a and b for SP. Write a simplified expression for SP.

5 The area of HEFG is equal to the area of 4 triangles and the square PQRS. Write an expression for the area of HEFG in terms of a and b. Simplify the expression.

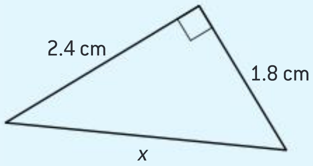
6 Find the area of HEFG in terms of c. Hence, by equating this to the expression you found in question 5, write an equation with a, b and c.

Key point

Pythagoras' theorem says that in a right-angled triangle $a^2+b^2=c^2$, where a and b are the lengths of the two perpendicular sides and c is the side opposite the right angle (the **hypotenuse**).



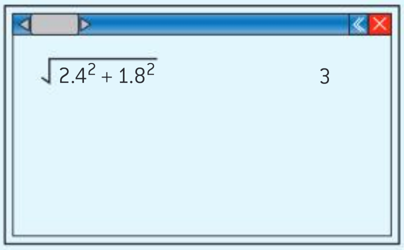
Example 1



Find the missing length x.

In the right-angled triangle:

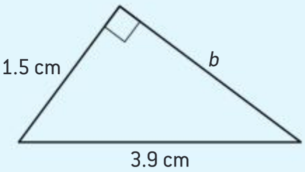
$$x^2=2.4^2+1.8^2$$
$$x=\sqrt{2.4^2+1.8^2}$$
$$x=3\text{ cm}$$



$\sqrt{2.4^2+1.8^2}$ 3

Enter the expression for the square root to get your result in a single step to avoid rounding errors.

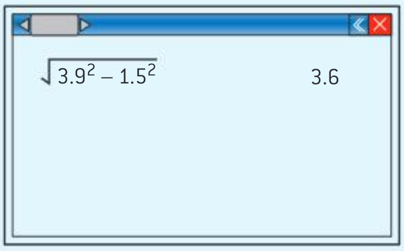
Example 2



Find the missing length b.

In the right-angled triangle:

$$1.5^2 + b^2 = 3.9^2$$
$$b^2 = 3.9^2 - 1.5^2$$
$$b = \sqrt{3.9^2 - 1.5^2}$$
$$b = 3.6 \text{ cm}$$



$\sqrt{3.9^2 - 1.5^2}$ 3.6

Hint

The hypotenuse is the longest side in a right-angled triangle. Check that the value you have for the hypotenuse is greater than the other two sides.

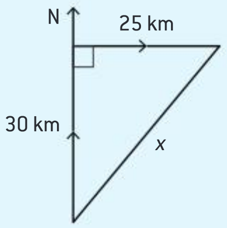
Pythagoras' theorem may be used to solve simple problems involving right-angled triangles.

Example 3

a A ship travels 30 km north and then a further 25 km east. Calculate the distance travelled by the ship from its starting position.

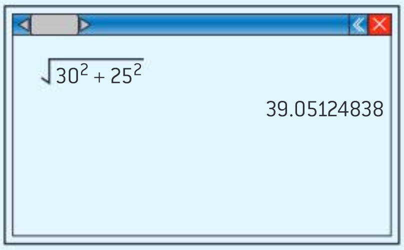
b A 2.4 m ladder rests, on horizontal ground, against a vertical wall. The ladder reaches 2.3 m up the wall. Calculate the distance from the bottom of the wall to the foot of the ladder.

a



The distance of the ship from its starting position is x.

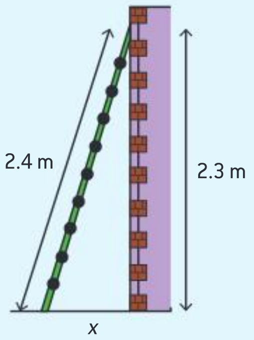
$$x^2 = 30^2 + 25^2$$
$$x = \sqrt{30^2 + 25^2}$$
$$x = 39.1 \text{ km (3 sf)}$$



$\sqrt{30^2 + 25^2}$

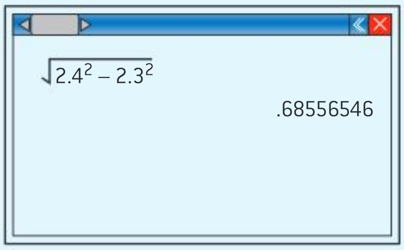
39.05124838

b



The distance from the foot of the ladder to the wall is x.

$$x^2 + 2.3^2 = 2.4^2$$
$$x^2 = 2.4^2 - 2.3^2$$
$$x = \sqrt{2.4^2 - 2.3^2}$$
$$x = 0.686 \text{ m}$$



$\sqrt{2.4^2 - 2.3^2}$

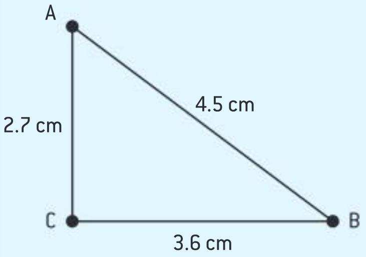
.68556546

Key point

The converse of Pythagoras' theorem states if $a^2 + b^2 = c^2$ then the triangle is right-angled.

The **converse** of a logical statement is obtained by switching the two parts of the statement around.

Example 4

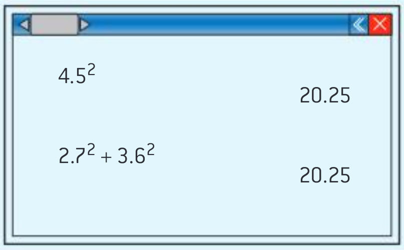


In triangle ABC, AB = 4.5 cm, AC = 2.7 cm and BC = 3.6 cm. Show that ABC is right-angled.

The longest side is AB. If ABC is right-angled then this will be the hypotenuse.

$$4.5^2 = 20.25$$
$$2.7^2 + 3.6^2 = 20.25$$

Since $AB^2 = AC^2 + BC^2$, ABC is right-angled at C.



4.5^2

20.25

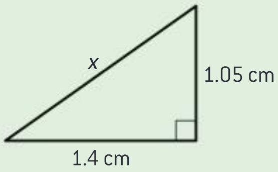
$2.7^2 + 3.6^2$

20.25

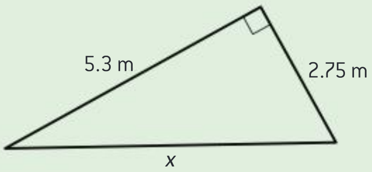
Exercise 6.1

1 In these questions find the length marked x.

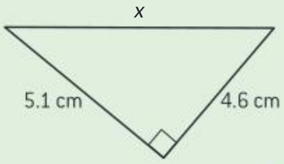
a



b

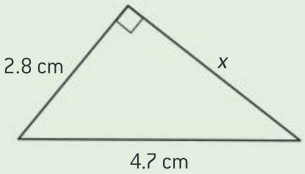


c

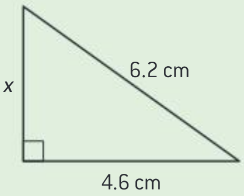


2 In these questions find the length marked x.

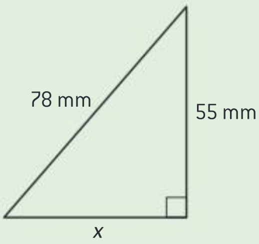
a



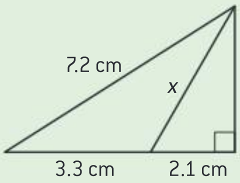
b



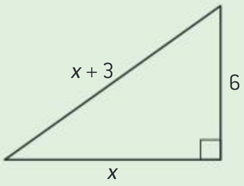
c



3 Find the length marked x.



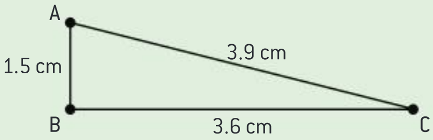
4 Form an equation in x and solve it to find the lengths of the sides of the triangle.



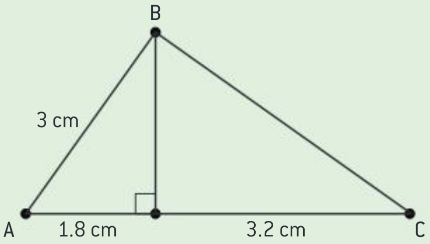
5 A plane flies 120 km West and then flies 220 km South. Calculate the distance the plane must fly to return to its starting point.

6 A ladder is 3.7 m long. If the foot of the ladder is placed on horizontal ground 1 m from a vertical wall, how high will the ladder reach?

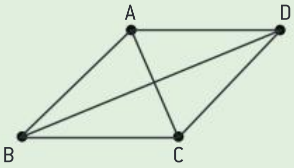
7 Show that the triangle ABC is right-angled.



8 Find the length BC and hence show that Δ ABC is right-angled at B.

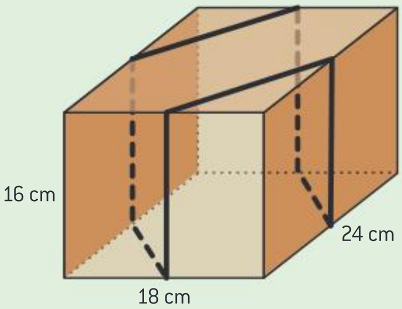


9 ABCD is a rhombus. The diagonals AC and BD are 4 cm and 8 cm. Find the length of the sides of the rhombus.



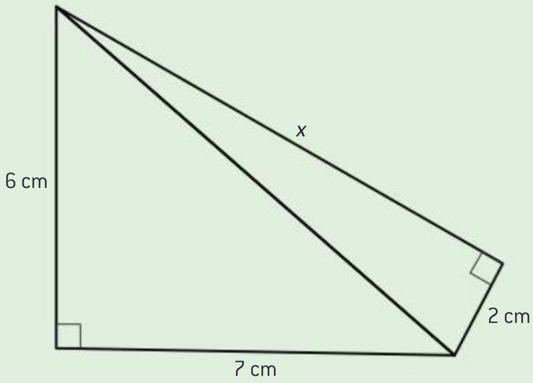
DP style Applications and Interpretation SL

10 A string is tied around a box so that the string crosses at the midpoints of the sides. The box is 24 cm long, 18 cm wide and 16 cm high. Find the length of the string.



DP style Analysis and Approaches SL

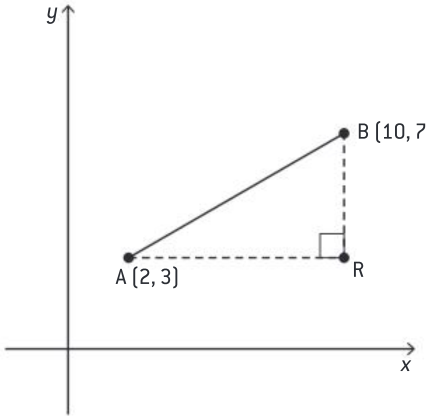
11 Find the length x.



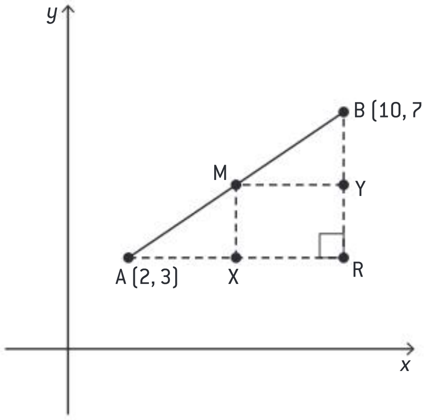
6.2 Mid-point of a line segment and the distance between two points

Investigation 6.2

A and B are two points with coordinates (2, 3) and (10, 7). A line is drawn parallel to the x-axis through A and another parallel to the y-axis through B. the lines meet at R to form a right-angled triangle ABR.



1 Write down the coordinates of the point R.



X is the midpoint of the line AR and Y is the midpoint of the line BR.

2 **a** Find the coordinates of X so that AX = XR.

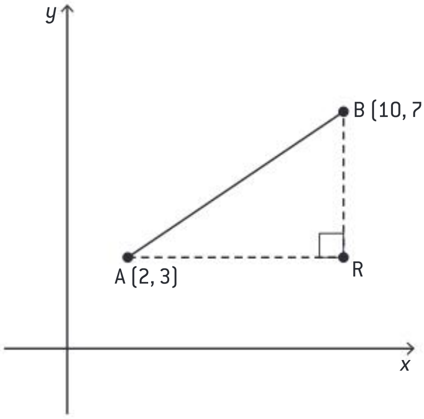
b Find the coordinates of Y so that BY = YR.

Draw lines through X and Y to meet at M on AB. MX is parallel to and equal to YR and MY is parallel to and equal to XR.

c Show that the triangles AMX and MBY are congruent hence deduce that AM = MB.

M is the midpoint of AB.

3 Write down the coordinates of the point M.



4 **a** The midpoint of 4 and 8, is given by $\frac{4+8}{2}=6$. Verify this method with some other numbers.

b If A is the point (x_1, y_1) and B is the point (x_2, y_2) write an expression in terms of x_1, x_2, y_1 and y_2 for the coordinates of M.

In the second part of this investigation you will look at the length of AB.

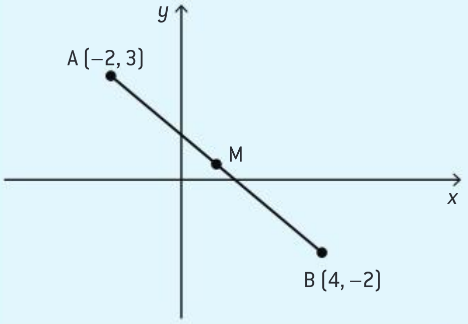
5 Write down the coordinates of R.

6 Find the lengths of AR and BR.

7 Δ ARB is a right-angled triangle. Use Pythagoras' theorem to find AB.

8 If A is the point (x_1, y_1) and B is the point (x_2, y_2) find the lengths of AR and BR and hence write an expression in terms of x_1, x_2, y_1 and y_2 for the length of AB.

Example 5



A is the point (–2, 3) and B is the point (4, –2).

Find the coordinates of the midpoint M of the line segment [AB] and the length AB.

$$M=\left(\frac{-2+4}{2},\frac{3-2}{2}\right)$$

$$=\left(1,\frac{1}{2}\right)$$

Take care with negative values.

$$AB=\sqrt{\left(4-(-2)\right)^2+(-2-3)^2}$$

$$=\sqrt{61}$$

$$=7.81\,(3\text{ sf})$$

Key point

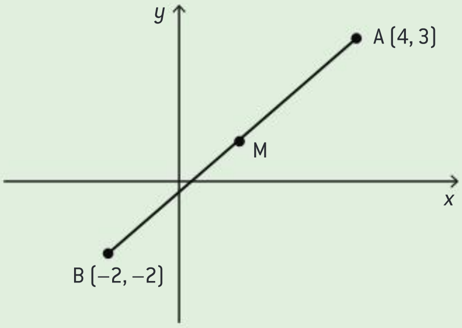
The midpoint of the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{1}{2}(x_1+x_2),\frac{1}{2}(y_1+y_2)\right)$

Key point

The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Exercise 6.2

1 A is the point (4, 3) and B is the point (–2, –2)



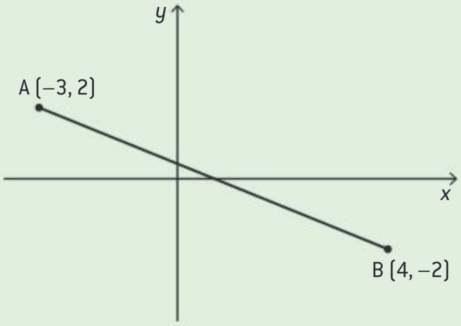
Find the midpoint, M of the line segment [AB]

2 Find the midpoint of the line joining these points:

a (1, 2) and (5, 6) **b** (–2, 6) and (3, 8)

c (3, –3) and (–2, –3) **d** (2, 5) and (–1, 0).

3 A is the point (–3, 2) and B is the point (4, –2)



Find the length AB.

4 Find the distance between these points:

a (3, 2) and (4, 3) **b** (–4, 1) and (1, 1)

c (6, –1) and (–4, 5) **d** (3, 3) and (–4, –3).

DP style **MAA and MAI HL**

5 The route of a train track lies along the line $y=2x+5$. A town is situated at a point with coordinates $(30,0)$. It is decided to build a new station as close as possible to the town. As a surveyor you are asked to provide the coordinates of where the station should be.

a Explain why the line from the proposed station to the town should make an angle of 90° with the track.

b Find the distance from the town to the point $(0,5)$, giving your answer in the form $\sqrt{a}$ where a is an integer.

Let the coordinates of the point on the train track closest to the town be (p,q).

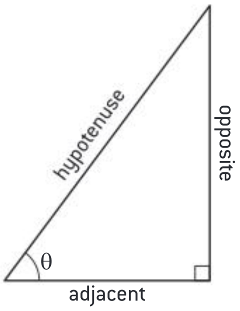
c Give an expression for q in terms of p.

d Use Pythagoras' Theorem to find the value of p and hence the coordinates of the new station.

Internal link

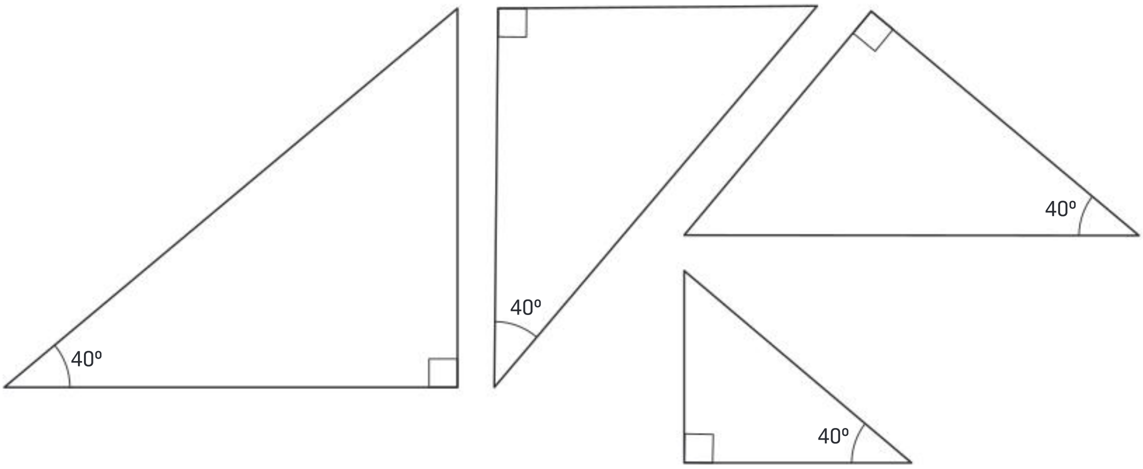
In Investigation 5.4, you investigated enlargements to show that they were similar. Two shapes are similar if corresponding lengths are in the same ratio. Corresponding angles will be the same. The shape will be the same, but a different size.

6.3 Right-angle trigonometry



One of the angles of the right-angled triangle is labelled θ (pronounced *theta*). The two sides are labelled **opposite** the angle and **adjacent** (next to) the angle. The third side is the hypotenuse. You can abbreviate these as opp, adj and hyp.

The triangles below are all right-angled and have an angle of 40°. For each triangle, measure, as accurately as possible, the length of the hypotenuse and the lengths of the sides that are opposite and adjacent to the 40° angle.



For each triangle calculate $\frac{\text{opp}}{\text{hyp}}$, $\frac{\text{adj}}{\text{hyp}}$ and $\frac{\text{opp}}{\text{adj}}$ and show that the corresponding ratios are equal.

Draw four other right-angled triangles accurately, all with the same angle. Measure the opposite, adjacent and hypotenuse and show that the corresponding ratios are the same as before.

The ratios of the sides of a right-angled triangle are called the trigonometric ratios.

There are three ratios (or fractions) of lengths of sides in this triangle. They are known as **sine**, **cosine** and **tangent**. The abbreviations commonly used for these ratios are sin, cos and tan.

Key point

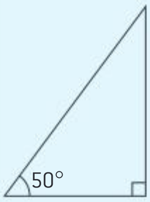
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$

You can find these ratios for any angle with your GDC. Since θ is one of the angles of a right-angled triangle $0° < \theta < 90°$. Some GDCs are set to work in radians by default and it is important to check which mode yours is in and set it to work in DEG before attempting any questions on trigonometry.

Find sin 40°, cos 40° and tan 40° with your GDC. Verify that the values obtained are the same as the ratios that you found by measuring the sides of the triangles.

Example 6



Find:

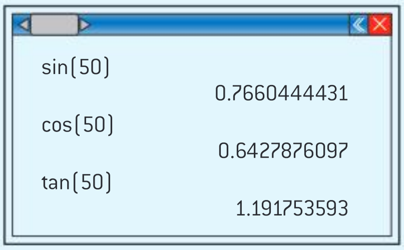
a sin 50° **b** cos 50° **c** tan 50°.

a sin 50° = 0.766 (3 sf)

b cos 50° = 0.643 (3 sf)

c tan 50° = 1.19 (3 sf)

Enter the sin, cos and tan functions in your GDC.



```
sin(50)
                0.7660444431
cos(50)
                0.6427876097
tan(50)
                 1.191753593
```

Finding lengths of sides in right-angled triangles

Because the ratios are the same for any right-angled triangle with a given angle, you can use them to find the length of one side of the triangle when we know another. You can also use them to find the angle in a right-angled triangle when you know two of the sides. You will need to choose which of the three ratios to use, depending on the combination of sides that you are working with.

"SOHCAHTOA" is a helpful way to remember these definitions.

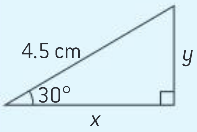
Key point

Use *Sine* when you have one of Opposite and Hypotenuse and want to know the other.

Use *Cosine* when you have one of Adjacent and Hypotenuse and want to know the other.

Use *Tangent* when you have one of Opposite and Adjacent and want to know the other.

Example 7



Find the missing lengths x and y.

x is adjacent to the angle so you should use cos

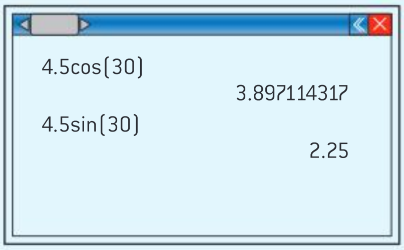
$$\frac{x}{4.5} = \cos 30°$$
$$x = 4.5\cos 30°$$
$$x = 3.90 \text{ cm (3 sf)}$$

y is opposite the angle so you should use sin

$$\frac{y}{4.5} = \sin 30°$$
$$y = 4.5\sin 30°$$
$$y = 2.25 \text{ cm (3 sf)}$$

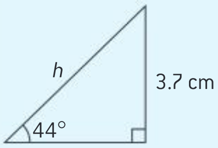
Rearrange to find x and to find y.

Enter the calculation directly into the GDC.



```
4.5cos(30)
                 3.897114317
4.5sin(30)
                        2.25
```

Example 8



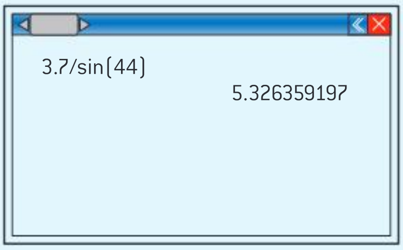
Find the length of the hypotenuse, *h*.

You have *opp* and want to find *hyp*, so use sine

$$\frac{3.7}{h} = \sin 44°$$
$$h \times \sin 44 = 3.7$$
$$h = \frac{3.7}{\sin 44°}$$
$$h = 5.33 \text{ cm} (3 \text{ sf})$$

Rearrange the equation to find *h*.

Enter the calculation directly into the GDC.

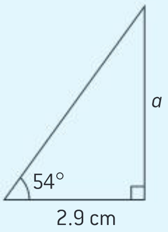


```
3.7/sin(44)
                    5.326359197
```

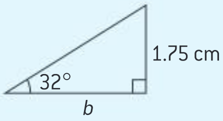
The examples you have seen so far, using sine and cosine, all involve the hypotenuse. When you are given the opposite and adjacent sides, you will use the tangent of the angle.

Example 9

a Find the length of the side marked *a*.



b Find the length of the side marked *b*.

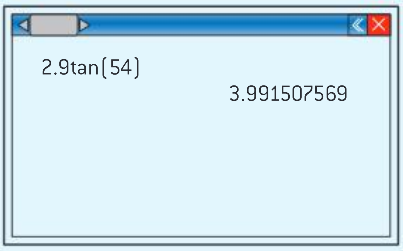


a Find the opposite side given the adjacent, using tan.

$$\frac{a}{2.9} = \tan 54°$$
$$a = 2.9 \tan 54°$$
$$a = 3.99 \text{ cm} (3 \text{ sf})$$

Rearrange the equation to find *a*.

Enter the calculation directly into the GDC.



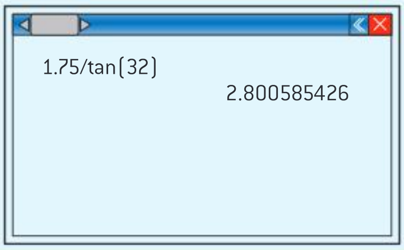
```
2.9tan(54)
                    3.991507569
```

b Find the adjacent side given the opposite, using tan.

$$\frac{1.75}{b} = \tan 32°$$
$$b \cdot \tan 32° = 1.75$$
$$b = \frac{1.75}{\tan 32°}$$
$$b = 2.80 \text{ cm } (3 \text{ sf})$$

Rearrange the equation to find *b*. This is similar to the rearrangement you used when finding the hypotenuse above.

Enter the calculation directly into the GDC.



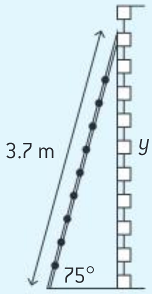
```
1.75/tan(32)
                    2.800585426
```

Example 10

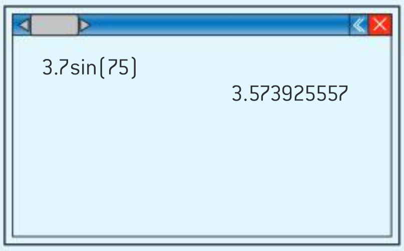
a A ladder should be at 75° to the horizontal to be safe. If my ladder is 3.7 m long, what height can I safely reach?

b A ship is sailing on a bearing of 156°. When it arrives at its destination it is 35 km south of its starting point. How far has it travelled?

a Let the height reached be y m.

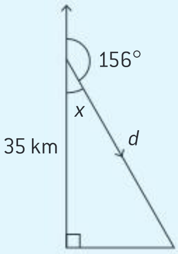


$$\frac{y}{3.7} = \sin 75^\circ$$
$$y = 3.7\sin 75^\circ$$
$$y = 3.57 \text{ m (3 sf)}$$

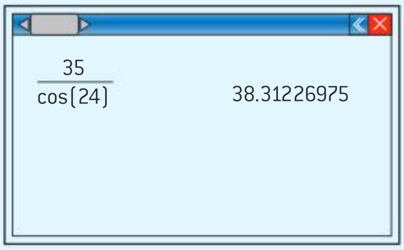


```
3.7sin(75)
                    3.573925557
```

b Let the distance travelled be d km.



$$\frac{35}{d} = \cos 24^\circ$$
$$d = \frac{35}{\cos 24^\circ}$$
$$d = 38.3 \text{ km (3 sf)}$$

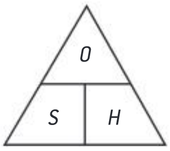


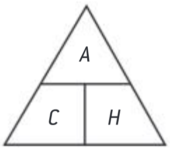
$$\frac{35}{\cos(24)} \qquad 38.31226975$$

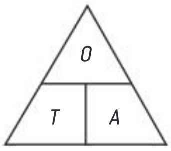
The angle x is $180 - 156 = 24^\circ$.

It is always useful to check that the hypotenuse is the longest side after your calculations. If not, then you have made a mistake.

These triangles will help you with rearranging the equations.







The diagrams show you graphically that

$$\sin = \frac{\text{opp}}{\text{hyp}} \qquad \cos = \frac{\text{adj}}{\text{hyp}} \qquad \tan = \frac{\text{opp}}{\text{adj}}$$

$$\text{opp} = \sin \times \text{hyp} \quad \text{adj} = \cos \times \text{hyp} \quad \text{opp} = \tan \times \text{adj}$$

$$\text{hyp} = \frac{\text{opp}}{\sin} \qquad \text{hyp} = \frac{\text{adj}}{\cos} \qquad \text{adj} = \frac{\text{opp}}{\tan}$$

Note

Remember you have already learned that “SOHCAHTOA” is a helpful way to remember these definitions.

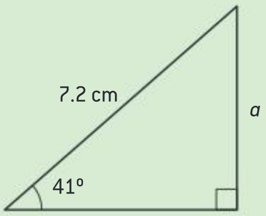
Exercise 6.3a

1 Find:

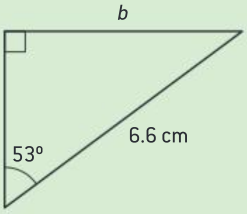
a sin 46° **b** cos 21° **c** tan 87°

2 Find the lettered lengths.

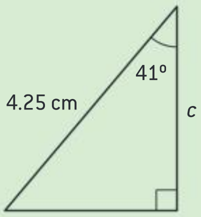
a



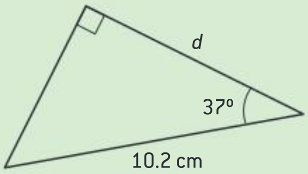
b



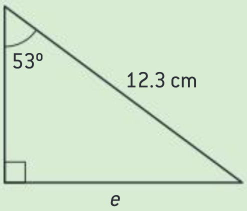
c



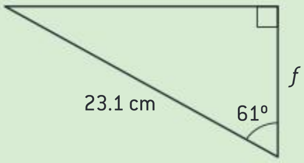
d



e

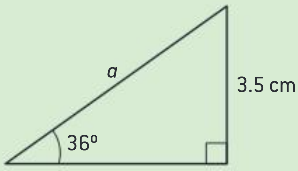


f

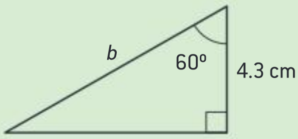


3 Find the lettered length.

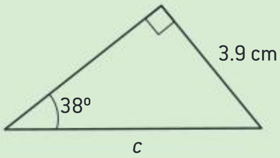
a



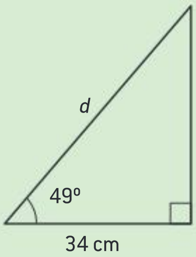
b



c

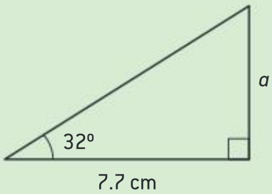


d

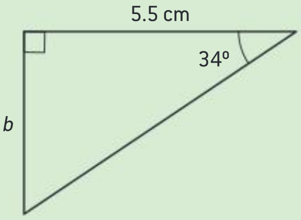


4 Find the lettered sides.

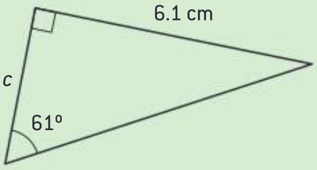
a



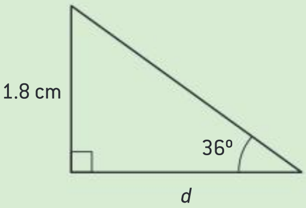
b



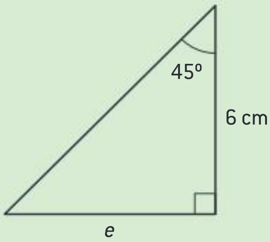
c



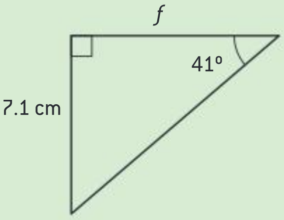
d



e



f



5 A plane Is flying on a bearing of 073°. If plane has travelled a distance 183 km North, how far East has it travelled?

6 A kite is flying so that its string is taut. The string is 42 m long and makes an angle of 68° with the ground. How high is the kite?

Finding angles in right-angled triangles

If you are given two sides of a right-angled triangle, then it is always possible to find the size of an angle. You find an angle from the trigonometric ratio with the inverse trigonometric functions. These functions are $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Example 11

Find θ using your GDC.

a $\sin\theta = 0.234$ **b** $\cos\theta = 0.897$ **c** $\tan\theta = 1.51$

a $\sin\theta = 0.234$

$\theta = \sin^{-1}(0.234)$

$\theta = 13.5°$ (3 sf)

b $\cos\theta = 0.897$

$\theta = \cos^{-1}(0.897)$

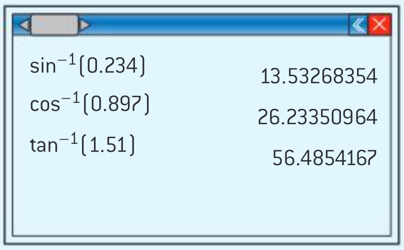
$\theta = 26.2°$ (3 sf)

c $\tan\theta = 1.51$

$\theta = \tan^{-1}(1.51)$

$\theta = 56.5°$ (3 sf)

Enter the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ functions in your GDC.

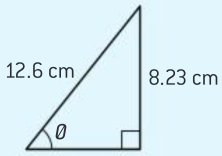


```
sin⁻¹(0.234)
                13.53268354
cos⁻¹(0.897)
                26.23350964
tan⁻¹(1.51)
                56.4854167
```

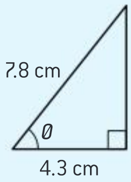
Example 12

Find θ using your GDC.

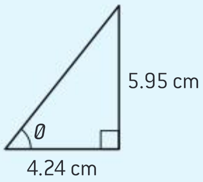
a



b



c



a $\sin\theta = \dfrac{8.23}{12.6}$

$\theta = \sin^{-1}\left(\dfrac{8.23}{12.6}\right)$

$\theta = 40.8°$ (3 sf)

You have opposite and hypotenuse, so use sine ratio.

Use $\sin^{-1}$ and enter the calculations directly into your GDC.

b $\cos\theta = \dfrac{4.3}{7.8}$

$\theta = \cos^{-1}\left(\dfrac{4.3}{7.8}\right)$

$\theta = 56.5°$ (3 sf)

You have the adjacent and hypotenuse, so use cos ratio.

c $\tan\theta = \dfrac{5.95}{4.24}$

$\theta = \tan^{-1}\left(\dfrac{5.95}{4.24}\right)$

$\theta = 54.5°$ (3 sf)

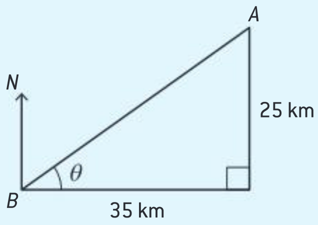
You have the opposite and adjacent, so use tan ratio.

Example 13

a Two radio beacons at A and B are used to guide planes to a landing strip. A is 25 km North of B and 35 km east. Calculate the bearing of A from B.

b A plank is leaning against a vertical wall. The plank is 2.1 m long and reaches 1.8 m up the wall. Calculate the angle between the plank and the horizontal ground.

a

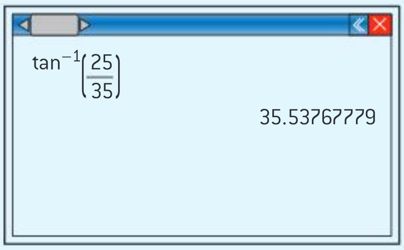


$$\tan\theta = \frac{25}{35}$$

$$\theta = \tan^{-1}\left(\frac{25}{35}\right)$$

$$\theta = 35.5° \text{ (3 sf)}$$

Calculate the angle θ.



$\tan^{-1}\left(\frac{25}{35}\right)$

35.53767779

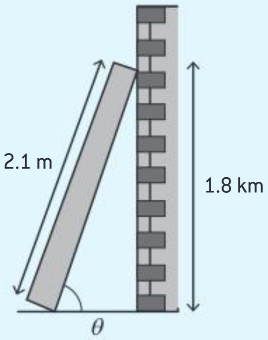
$90 - 35.5 = 54.5$

The bearing is 054.5°.

Find the bearing.

b

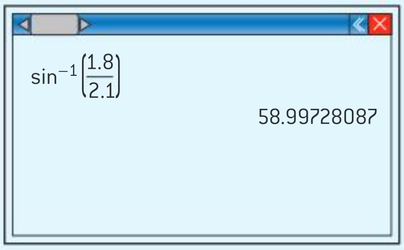
Calculate the angle θ.



$$\sin\theta = \frac{1.8}{2.1}$$

$$\theta = \sin^{-1}\left(\frac{1.8}{2.1}\right)$$

$$\theta = 59.0° \text{ (3 sf)}$$



$\sin^{-1}\left(\frac{1.8}{2.1}\right)$

58.99728087

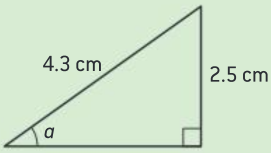
Exercise 6.3b

1 Calculate the angles.

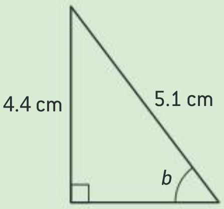
a $a = \sin^{-1} 0.793$ **b** $b = \cos^{-1} 0.184$ **c** $c = \tan^{-1} 0.891$

2 Find the lettered angles.

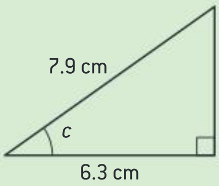
a



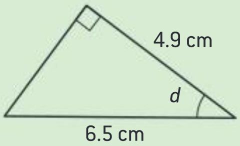
b



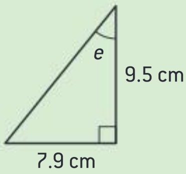
c



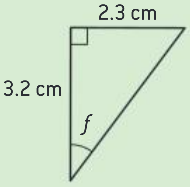
d



e



f



3 A ship sails to a harbour that is 54 km south and 44 km west of the starting point. Find the bearing of the harbour from the starting point.

4 A children's slide is 1.9 m long. If the height of the slide is 1.4 m, calculate the angle between the slide and the horizontal.

6.4 Problem solving with right-angled triangles

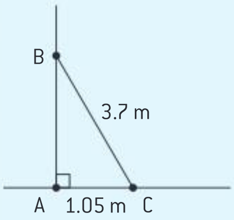
Problems with right-angled triangles may involve the use of Pythagoras' theorem and/or trigonometry. In problems that need several steps to obtain a solution, avoid rounding results until you reach the final answer. Either keep the full value that your calculator stores or, at the very least, record intermediate results to one or two extra significant figures.

Example 14

A post 3.7 m long is placed at an angle against a vertical fence. The base of the post is 1.05 m from the fence on horizontal ground.

a Calculate the height of the point where the post meets the fence.

b Safety regulations say that the post must make an angle of at least 70° with the ground. Is the post safe?



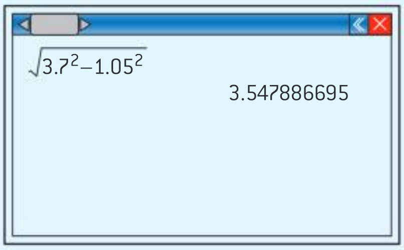
Draw a diagram of the fence, the post and the ground. A vertical fence means it is at right angles to the ground.

a $AB^2 + 1.05^2 = 3.7^2$

$$AB = \sqrt{3.7^2 - 1.05^2}$$

$$AB = 3.55 \text{ m (3 sf)}$$

Use Pythagoras' theorem to find the length AB.



$\sqrt{3.7^2-1.05^2}$

3.547886695

b $\cos A\hat{C}B = \dfrac{1.05}{3.7}$

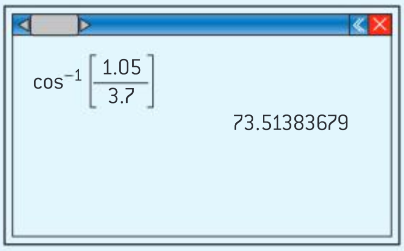
$$A\hat{C}B = \cos^{-1}\frac{1.05}{3.7}$$

$$A\hat{C}B = 73.5° \text{ (3 sf)}$$

73.5° > 70°, so the post is safe.

Use cosine to find angle $A\hat{C}B$.

Use the lengths you were given rather than the one you have just calculated, in case of any errors.

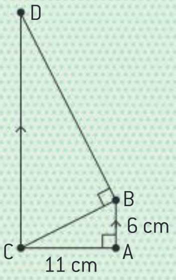


$\cos^{-1}\left[\frac{1.05}{3.7}\right]$

73.51383679

Exercise 6.3c

1 In the diagram, $C\hat{B}D$ and $C\hat{A}B$ are right-angles. AB and CD are parallel.

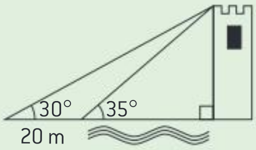


Calculate the size of the angle $C\hat{D}B$.

DP style **Analysis and Approaches HL**

2 A surveyor is trying to find the height of a vertical tower on the other side of a river. Angle of elevation is the angle above the horizontal of a distant object. The surveyor measures the angle of elevation of the top of the tower, which is 30°. He then moves right to the edge of the river, which is 20 m closer to the tower, and measures the angle of elevation again. It is now 35°.

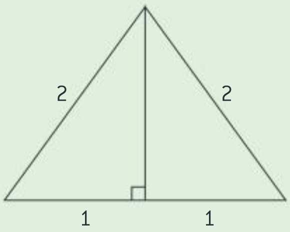
Let h be the height of the tower and w the width of the river. Write down two equations in h and w and solve these to find the height of the tower



Calculate the height of the tower.

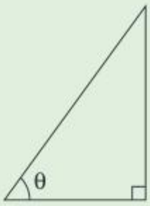
DP style **Analysis and Approaches**

3 Consider the right-angled triangle formed by dividing an equilateral triangle of side length 2 into two congruent triangles as shown.



a Use the diagram to prove that tan 60° = $\sqrt{3}$ and find the exact value of tan 30°.

Consider the diagram shown:



Let $\tan\theta = t$

b Find an expression for tan (90° – θ) in terms of t.

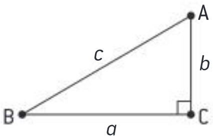
c Hence solve:

i $\tan\theta + 3\tan(90° - \theta) = 2\sqrt{3}$

ii $\tan\theta + \tan(90° - \theta) = \dfrac{4}{\sqrt{3}}$

Chapter summary

☐ Pythagoras' theorem says that in a right-angled triangle, if the two sides at right-angles are a and b, and if the side opposite the right angle (the **hypotenuse**) is c then $a^2 + b^2 = c^2$.



☐ The converse of Pythagoras' theorem states if $a^2 + b^2 = c^2$ then the triangle is right-angled.

☐ The midpoint of the line joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$

☐ The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

☐ "SOHCAHTOA" is a helpful way to remember these definitions.

☐ These equations can be used to find:

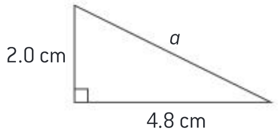
  ☐ a side, if you know the angle and one other side

  ☐ an angle, if you know the lengths of two sides.

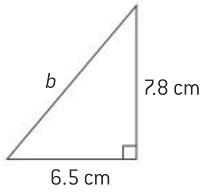
Chapter 6 test

1 Find the lettered lengths.

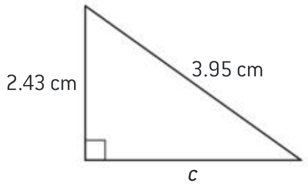
a



b



c



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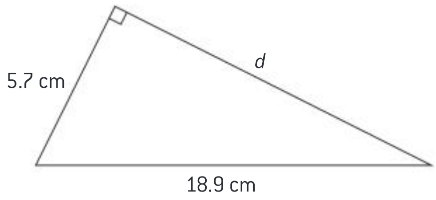

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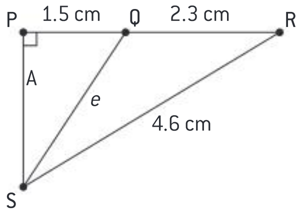
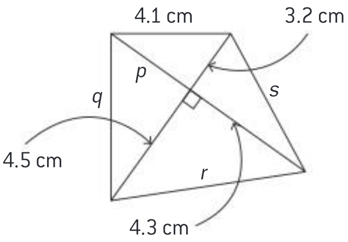


2 A ship sails to a port that is 26 km South and 44 km West of its starting point. How far does the ship travel?

3 Find the lengths of the sides marked p, q, r and s.



4 Find the midpoint of the line joining (5, 8) and (1, 2).

5 Find the distance between the points (–3, 6) and (4, –2).

6 Use your GDC to find the values of:

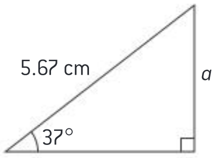
a sin 34° **b** cos 52° **c** tan 13°

d $\sin^{-1}$ 0.235 **e** $\cos^{-1}$ 0.775

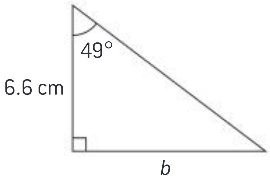
f $\tan^{-1}$ 0.653.

7 Find the lettered lengths.

a



b



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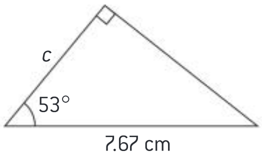

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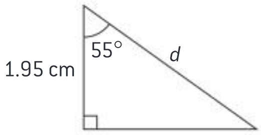

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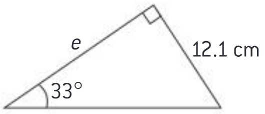

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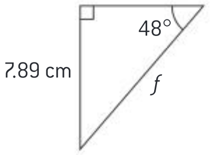
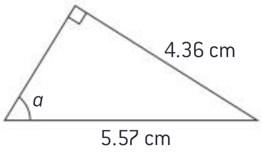


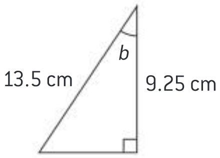
8 A 3.3 m ladder leans against a vertical wall. The angle between the ladder and the horizontal ground it stands on is 76°. How high up the wall does the ladder reach.

9 Find the lettered angles.

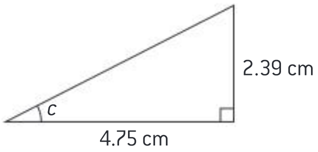
a



b



c



10 A plane travels to a point that is 245 km north and 367 km east. What is the bearing of the destination?

11 A plane flies 230 km on a bearing of 127°. Calculate:

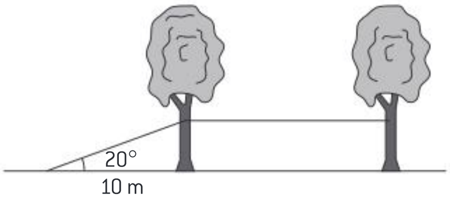
a the total distance travelled east

b the total distance travelled south.

DP style **Applications and Interpretation SL**

12 Some workers are constructing an aerial walkway between two trees on horizontal ground. They put up a ramp to the walkway that is 10 m from the base of a tree and at an angle of 20° to the horizontal.

It is decided that the ramp is too steep and it must be remade so that it is at an angle of 15°. Calculate how much further away from the foot of the tree the ramp must start.



DP style **Analysis and Approaches SL**

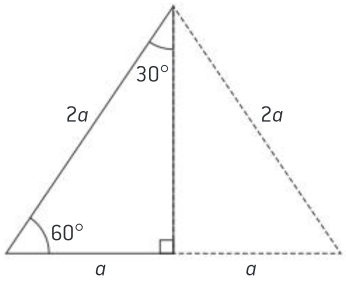
13 If the sides of a rhombus are 4.2 cm long and one of the diagonals is 5.7 cm, find the angles in the rhombus.

Internal link

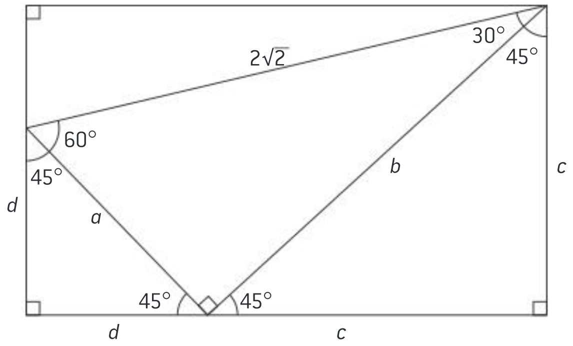
Recall the properties of a rhombus, which you studied in chapter 5.

DP style **Applications and Interpretation HL**

14 A right-angled triangle with angles 30°, 60° and 90° can be thought of as half an equilateral triangle, and hence the shorter side adjacent to the 60° angle will be half the hypotenuse.



Consider the rectangle shown (Ailles rectangle). This will be used to calculate the exact values of the trigonometric ratios for 15° and 75°.



a Write down the length *a*.

b Use Pythagoras' theorem to find the length *b*.

c Use Pythagoras' theorem to find the lengths:

i *c* **ii** *d*.

d Show the angles 15° and 75° on the diagram.

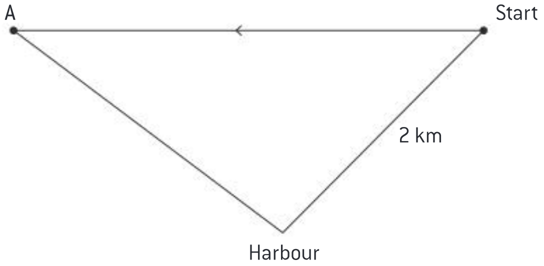
e Hence find the exact values of:

i cos 15° **ii** tan 75°.

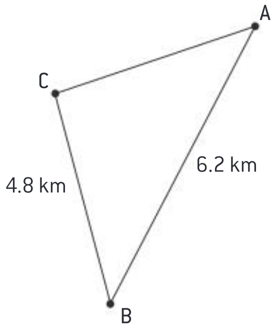
DP style **Applications and Interpretation HL**

15 a The start of a yacht race is 2 km from a harbour. The bearing of the harbour from the start is 220°.

From the start of the race a yacht sails west until the harbour is on a bearing of 105°. By dividing the triangle shown into two right-angled triangles, find the distance travelled during this part of the race.



b In the second stage of the race the yacht sails 6.2 km from A to B and then turns through an angle of 135° before sailing 4.8 km to C.



i Write down the angle ABC.

ii Find the distance from C back to A.

Modelling and investigation

DP ready **Approaches to learning**

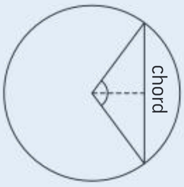
Critical thinking: Analysing and evaluating issues and ideas

Organization skills: Managing time and tasks effectively

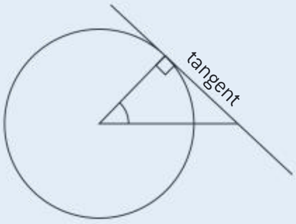
DP ready **International-mindedness**

In early trigonometry, calculations were done with the aid of tables of values of the trigonometric ratios. The earliest such table, equivalent to a table of values of sines, was made by Hipparchus in Greece around 140 BC. The term sine, cosine and tangent come from Latin words. The learning of ancient Greece passed through India and the Arab world, reappearing in Western Europe many centuries later.

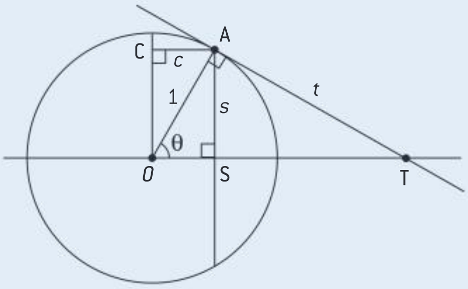
The term for *sine* came from the Latin word for a bay and it was used to describe a chord of a circle, that is a line joining two points on its circumference. Originally the length of this chord was used, but later, just half the length of the chord was used to define the sine.



The term *tangent* was from the Latin for touch and the tangent to a circle is a line that touches the circle at a point. The tangent and the radius at the point of contact are at right-angles.



1 In the following diagram, show that the sides marked *s*, *c* and *t* are equal to the sine, cosine and tangent of the angle θ. The circle is a *unit circle*, that is its radius is 1.

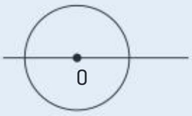


The sine is, in fact, half the length of the chord. The origins of the term cosine have to do with its relation to the complementary angle.

In the diagram, the angle alongside θ is $90° - \theta$, the complementary angle and the cosine is the sine of the complementary angle.

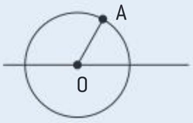
2 Using ruler, compasses and a pencil or, if you have access to an online geometry package, using Desmos or Geogebra, construct the diagram above. You can also use some GDCs to draw this construction.

a Begin by drawing a circle and a horizontal line through the point O.

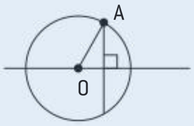


The radius of the circle does not matter as you will scale the diagram later.

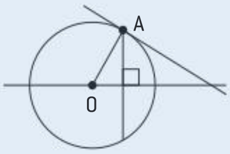
b Draw a line segment from O to a point A on the circumference.



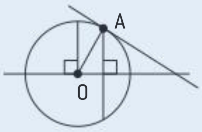
c Construct a *chord* from A perpendicular to the horizontal line.



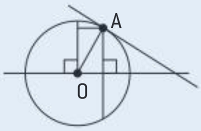
d Construct a *tangent* to the circle at A.



e Construct a *radius* from O that is perpendicular to the horizontal line.



f Construct a line segment from A perpendicular to the chord.



g If you are working with a geometry package, check your diagram by moving the point A around the circle. (Do not go beyond the horizontal line or the radius you drew). If you have drawn the figure correctly the lines will move with the point.

h Use the measurement tool to measure OA.

i Use the measurement tool to measure the angle marked θ.

j Use the measurement tool to measure the lengths marked *s*, *c* and *t*.

k To scale your results to those you would get with a *unit circle*, divide your values of *s*, *c* and *t* by the length of OA.

l Compare your results with the sin, cos and tan of the angle in your construction.

m Move the point A. The measurements should all change. Divide the new values by OA and compare these results with the sin, cos and tan of the new angle.

7

Volumes and areas of 2- and 3-dimensional shapes

Learning outcomes

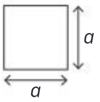
In this chapter you will learn about:

- The circle, its centre and radius, area and circumference. The terms diameter, arc, sector, chord, tangent and segment
- Perimeter and area of plane figures
- Familiarity with three-dimensional shapes (prisms, pyramids, spheres, cylinders and cones)
- Volumes and surface areas of cuboids, prisms, cylinders, and compound three-dimensional shapes

Key terms

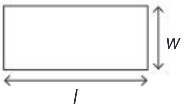
- Centre
- Radius
- Circumference
- Diameter
- Arc
- Sector
- Chord
- Tangent
- Segment
- Irrational
- Perimeter
- Area
- Polyhedron
- Prism
- Cuboid
- Cylinder
- Pyramid
- Cone
- Sphere
- Vertex
- Edge
- Face

7.1 Perimeter and area of plane figures



The sides of a square have length a. The perimeter of a square of side a is $4a$. The area is a^2.

Rectangles



The length of a rectangle is l and its width is w.

You find its perimeter by adding the lengths of each of its sides: $l + w + l + w = 2l + 2w = 2(l + w)$.

Example 1

Find the perimeter and area of a rectangle measuring 3 cm by 5 cm.

$$\begin{aligned} \text{Perimeter} &= 2(l + w) \\ &= 2(3 + 5) \\ &= 2 \times 8 \\ &= 16 \text{ cm} \\ \text{Area} &= l \times w \\ &= 3 \times 5 \\ &= 15 \text{ cm}^2 \end{aligned}$$

It is usually best to draw a diagram to help you.

Key point

The **perimeter** of a two-dimensional figure is the distance around its boundary. Perimeter is a measure of length so it is measured in cm, m, or km etc.

Key point

The **area** of a two-dimensional figure is a measurement of the amount of space inside its boundary. Area is measured in square units, such as cm^2.

Key point

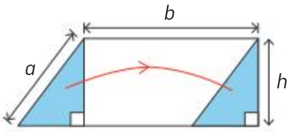
The perimeter of a rectangle is $2(l + w)$.

Key point

The area of a rectangle is lw.

Parallelograms

The sides of a parallelogram are a and b.



Take a right-angled triangle from one side and move it over to the other side. This makes a rectangle with the same area as the parallelogram. The perpendicular height of the triangle is h, and so the height of the rectangle is h.

The area of the rectangle is bh where h is the perpendicular height of the parallelogram.

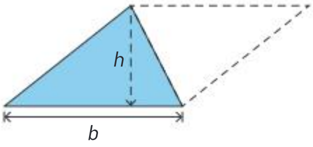
Key point

The area of the parallelogram is bh

Triangles

The diagonal of a parallelogram splits it into two identical triangles.

The perimeter of the triangle is the sum of the lengths of its sides.



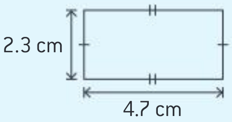
Key point

The area of the triangle is $\frac{1}{2}bh$.

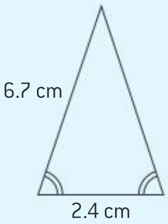
Example 2a

Find the perimeters of these shapes.

a



b



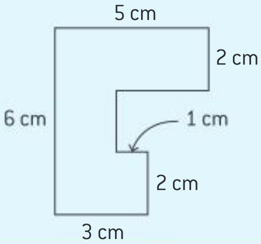
a 2.3 + 4.7 + 2.3 + 4.7 = 14 cm **b** 2.4 + 6.7 + 6.7 = 15.8 cm	Missing side is 6.7 cm (isosceles triangle).

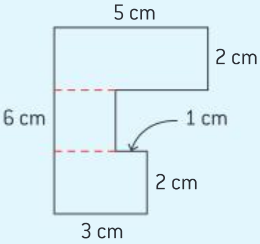
Example 2b

For this compound shape, find:

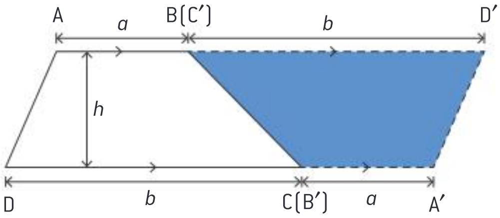
a the perimeter

b the area.



a 3 + 6 + 5 + 2 + 3 + 2 + 1 + 2 = 24 cm	Missing sides are 3 cm and 2 cm.
b A = (5 × 2) + (2 × 2) + (3 × 2) = 10 + 4 + 6 = 20 cm^2	Divide the composite shape up into shapes you know how to find the area of. 

Trapezoid



If you fit two trapezia together as shown, you get a parallelogram.

Area of two trapezia = $(a + b)h$ (using the formula for area of parallelogram).

Therefore area of one trapezoid is $A = \frac{1}{2}(a + b)h$

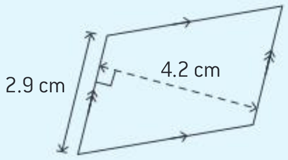
Key point

The area of a trapezoid is $\frac{1}{2}(a + b)h$.

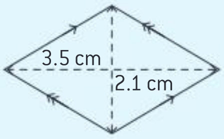
Example 3

Find the areas of these shapes.

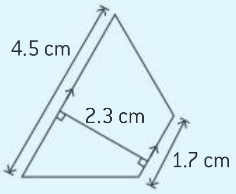
a



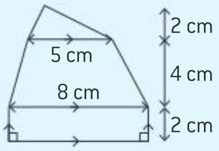
b



c



d



a Area = $b \times h$

$= 2.9 \times 4.2 = 12.2$ cm^2 (3 s.f.)

Two pairs of parallel sides mean this is a parallelogram.

b Area $= 4\left(\frac{1}{2} \times 3.5 \times 2.1\right)$

$= 14.7$ cm^2

The shape consists of 4 identical right-angled triangles with base 3.5 cm and height 2.1 cm.

c Area $= \frac{1}{2}(a + b)h$

$= \frac{1}{2}(4.5 + 1.7) \times 2.3$

$= 7.13$ cm^2

One pair of parallel sides mean this is a parallelogram.

d Area $= (2 \times 8) + \left(\frac{1}{2}(5 + 8) \times 4\right) + \left(\frac{1}{2} \times 5 \times 2\right)$

$= 16 + 26 + 5$

$= 47$ cm^2

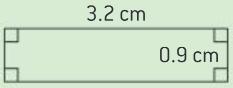
This is a *compound* shape, made from a rectangle, a trapezoid and a triangle.

If you look at formulae for finding an area, for example $l \times w$, πr^2 or ½ × base × height, they all involve multiplying one length by another. The base units for area are m^2.

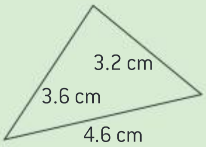
Exercise 7.1

1 Find the perimeters of these shapes.

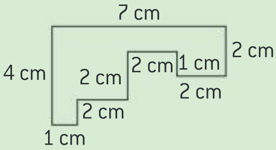
a



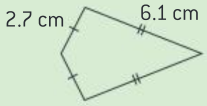
b



c

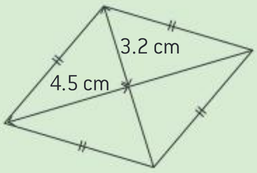


d

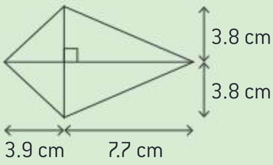


2 Find the areas of these shapes.

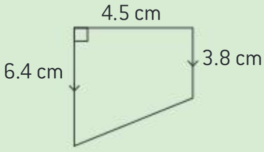
a



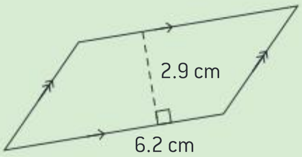
b



c

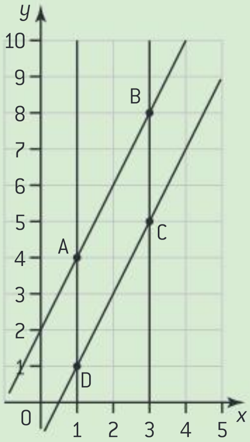


d



DP style **Analysis and Approaches HL**

3 The diagram shows the lines $y = 2x - 1$, $y = 2x + 2$, $x = 1$ and $x = 3$. The points of intersection are shown as A, B, C and D.

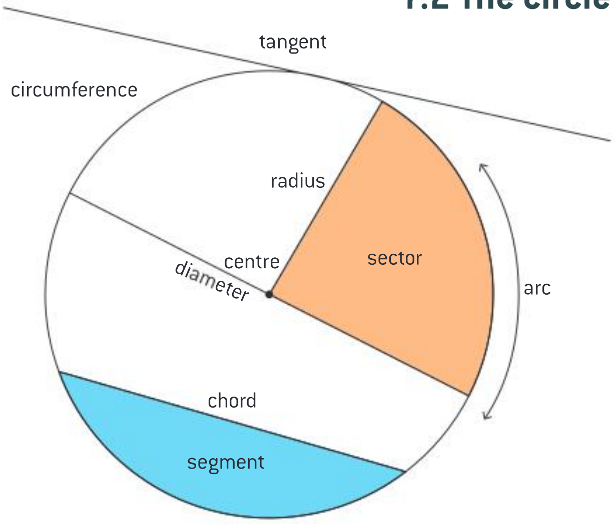


a Using [AD] as the base find the area of the parallelogram ABCD.

b Find the length of [AB]

c Use your answers to **a** and **b** to find the shortest distance (the perpendicular distance) between the lines $y = 2x - 1$ and $y = 2x + 2$

7.2 The circle



DP ready **Theory of knowledge**

The ratio of the circumference of a circle to its diameter has been used in mathematics for centuries. The earliest recorded reference is in the Rhind Papyrus and is an example of Ancient Egyptian mathematics. Other civilisations have their own references. There is plenty of archaeological evidence of the building of circular structures in many cultures around the world.

π is an **irrational** number. It cannot be written as an exact value. Mathematicians have calculated increasingly accurate approximations for π over the centuries. To 3 significant figures $\pi = 3.14$. For calculation you should always use the built-in value stored in your GDC which is 3.141592654… to provide sufficient accuracy.

Since the diameter (d) is twice the radius (r) $d = 2r$ and $C = 2\pi r$

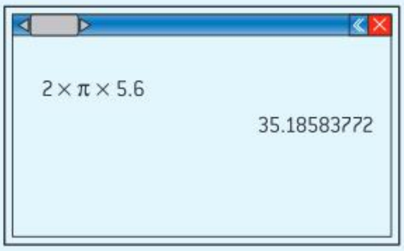
You can also write the area of the circle in terms of the radius. $A = \pi r^2$

Example 4

Find the circumference and area of a circle with a radius of 5.6 cm. Give your answers to 3 s.f.

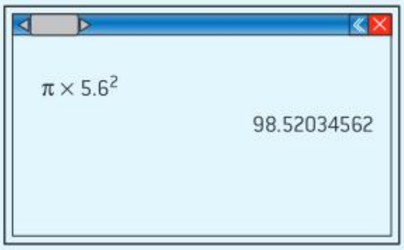
$$\begin{aligned}\text{Circumference} &= 2\pi r\\ &= 2 \times \pi \times 5.6\\ &= 35.2\text{ cm}\end{aligned}$$

$$\begin{aligned}\text{Area} &= \pi r^2\\ &= \pi \times 5.6^2\\ &= 98.5\text{ cm}^2\end{aligned}$$



$2 \times \pi \times 5.6$

35.18583772



$\pi \times 5.6^2$

98.52034562

Example 5

Find the radii of circles with:

a circumference = 44 cm **b** area = 60 cm^2

a

$$\begin{aligned}\text{Circumference} &= 2\pi r\\ 44 &= 2\pi r\\ r &= \frac{44}{2\pi}\\ r &= 7.00\text{ cm}\\ &\quad(3\text{ s.f.})\end{aligned}$$

to solve the equation for r, divide by 2π

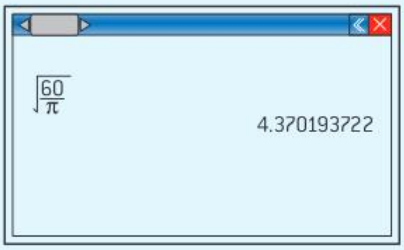
$\frac{44}{2\pi}$

7.002817496

b

$$\begin{aligned}\text{Area} &= \pi r^2\\ 60 &= \pi r^2\\ r^2 &= \frac{60}{\pi}\\ r &= \sqrt{\frac{60}{\pi}}\\ r &= 4.37\text{ cm (3 s.f.)}\end{aligned}$$

To solve the equation, divide by π then take square roots.



$\sqrt{\frac{60}{\pi}}$

4.370193722

Key point

A **circle** is a two-dimensional shape made by drawing a curve that is always the same distance from a point called the **centre**.

The distance around the edge of the circle is the **circumference**. This is the same as its perimeter.

The part of the circumference between two points on the circle is an **arc**.

A straight line joining two points on the circumference is a **chord**. If a chord passes through the centre, it is a **diameter**.

The **radius** is the distance from the centre to the circumference of the circle.

A chord cuts off an area called a **segment** and two radii cut off an area called a **sector**.

A straight line that touches the circumference at a point is a **tangent**.

The angle between a tangent and the radius at the point of contact is 90°.

Key point

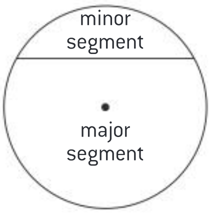
The ratio of a circle's circumference (C) to its diameter (d) is π. $\frac{C}{d} = \pi$

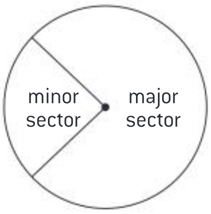
Key point

Circumference of a circle $C = 2\pi r$

Area of a circle $A = \pi r^2$

You can divide circles into major and minor segments, sectors and arcs. When these are equal, you have divided the circle into two semicircles.



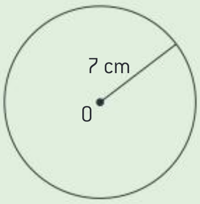


Exercise 7.2

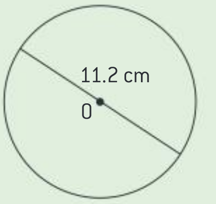
In each question, you should give answers to 3 s.f. where appropriate.

1 Find the circumference of these circles.

a



b

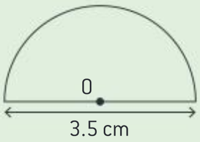


Hint

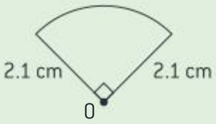
In part b, first find the radius of the circle.

2 Find the perimeters of these shapes.

a

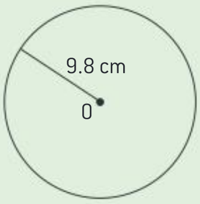


b

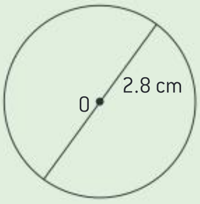


3 Find the area of these circles.

a

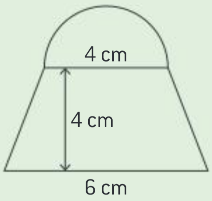


b

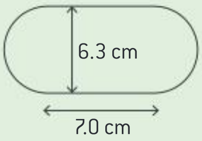


4 Find the areas of these shapes.

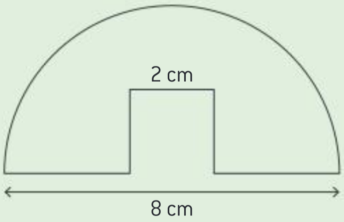
a



b



5 A compound shape is made from a semicircle with a square hole cut out. The shape is symmetrical. Find the area and perimeter of the shape.



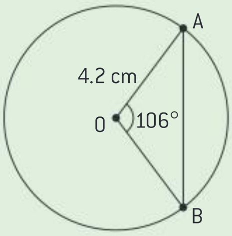
Hint

To find the perimeter, calculate all the individual lengths first. To find the area, subtract the area of the square from the area of the semicircle.

6 Find the radii of circles with:

a circumference = 32.5 cm **b** area = 24.5 cm^2.

7 Calculate the length of the chord AB. (Hint: split the triangle AOB into two right-angled triangles).



8 A tangent is drawn to a circle of radius 5 cm from a point A, 13 cm from its centre. Calculate the distance from A to the point where the tangent touches the circle.

Hint

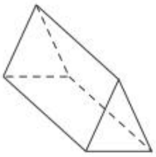
The angle between a radius and a tangent is 90°.

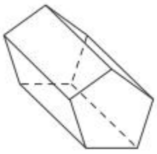
7.3 Three-dimensional shapes

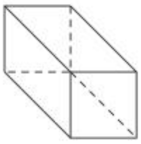
A 3-dimensional solid with straight sides is a **polyhedron**.

A prism is a 3-D solid with a constant cross-section (that is, if you cut into the solid at any point along its length, you will get the same shape as the one on the end face).

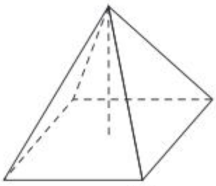
These solids are prisms:

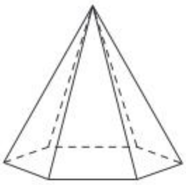

triangular prism


pentagonal prism

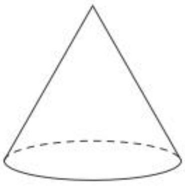

cuboid


cylinder

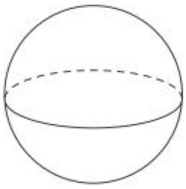

square-based pyamid


hexagonal-based pyamid

A **pyramid** is a polyhedron that has a base which gives it its name and triangular sides that meet at the apex.


cone

A cone has a circular base and a curved surface.


sphare

A **sphere** is a three-dimensional shape so that every point on its surface is the same distance from the centre.

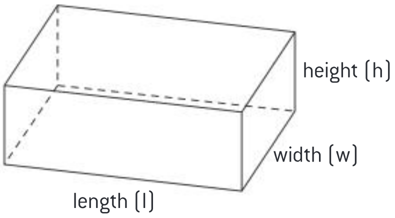
DP ready **International-mindedness**

For thousands of years, the largest structures in the world were pyramids. When you hear of pyramids, you tend to think of those in Egypt, but they are in many other parts of the world as far apart as China, India, Mexico, Peru, Nigeria and Indonesia. Early examples were often ziggurats, which are stepped pyramids, or even just mounds of earth and rock. Later, stone masons developed the art of cutting the stones on the outer surface to create a smooth pyramid. The existence of these ancient monuments has prompted people to speculate about the intervention of aliens in their building. However, the distribution of the weight of such buildings, with the majority of material nearer the ground and less weight pushing down from higher up allowed stable, very large structures to be built, often taking centuries to be completed as they were extended over the years.

Note

The pyramids and cone shown here are *right-pyramids* and *right-cones*, that is, the apex is directly above the centre of the base.

Volumes and surface areas of three-dimensional shapes



Key point

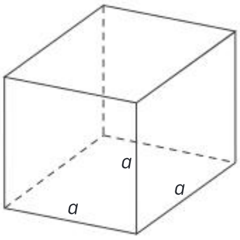
Volume is the amount of three-dimensional space something takes.

Key point

Surface area is the total area of the surface of a three-dimensional object.

The surface area of a cuboid is the sum of the areas of each of the faces.

Since a cube is a cuboid with equal length, width and height, say $l = w = h = a$



Key point

Volume of a cuboid, $V = lwh$.

Key point

Surface area of a cuboid, $S = 2lw + 2wh + 2hl$.

Then $V = a^3$ and $S = 6a^2$

Examples of formulae for volume are $l \times w \times h$ and $\pi r^2 h$. These formulae involve multiplying three lengths together. The base units for volume are m^3.

Internal link

Recall from chapter 3 that $1\ cm^3 = 1$ ml.

Exercise 7.3a

1 A cuboid shaped box measures 1.2 m by 35 cm by 38 cm. Find its volume in m^3 and its capacity in l.

2 Find the volume of a cuboid shaped box that measures 2.3 m by 0.45 m by 0.35 m in cm^3. Find also the surface area of the box in cm^2.

DP style **Applications and Interpretation SL**

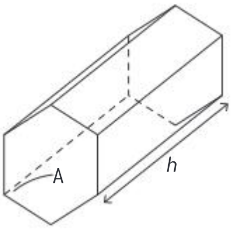
3 Iresh sees an advertisement for a cuboid-shaped freezer with a capacity of 139 litres. Its dimensions are 84.2 × 69.7 × 55.7 cm. What is the difference between the volume of the freezer and its storage capacity?

Volume of prisms

Key point

The volume of a prism, $V = Ah$.

The volume of any prism is the cross-sectional area × height. A cuboid, a cube and a cylinder are special cases of this.

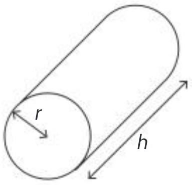


The surface area of a prism is 2 × cross-sectional area + perimeter of the cross-section × height.

Key point

The volume of a cylinder, $V = \pi r^2 h$.

In the same way the volume of a cylinder is the cross-sectional area × height.



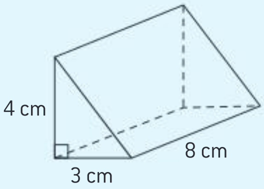
Key point

The surface area of a cylinder, $S = 2\pi r^2 + 2\pi rh$.

The surface area of a cylinder is given by 2 × area of the circular ends + circumference × height.

Example 6

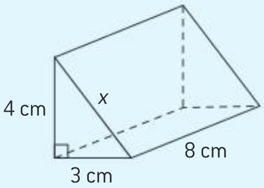
a A triangular prism has a cross-section which is a right-angled triangle with its shortest sides equal to 3 cm and 4 cm. The prism is 8 cm long.



Find the surface area and volume of the prism.

b A cylinder has the same volume and length as the prism. Find, correct to 3 significant figures, the radius of the cylinder and its surface area.

a



Let the third side of the triangle be x.

$$x^2 = 3^2 + 4^2$$
$$x = \sqrt{3^2 + 4^2}$$
$$x = 5 \text{ cm}$$

By Pythagoras' theorem.

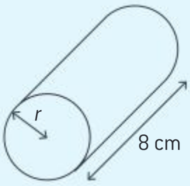
$$S = 2 \times \frac{1}{2} \times 3 \times 4 + 8(3 + 4 + 5)$$
$$S = 108 \text{ cm}^2$$

Surface area = 2 × area of triangle + perimeter of the triangle × height

$$V = \frac{1}{2} \times 3 \times 4 \times 8$$
$$V = 48 \text{ cm}^3$$

Volume = area of triangle × height

b



$$\pi \times r^2 \times 8 = 48$$
$$r^2 = \frac{48}{8\pi}$$
$$r = \sqrt{\frac{48}{8\pi}}$$
$$r = 1.38 \text{ cm}$$

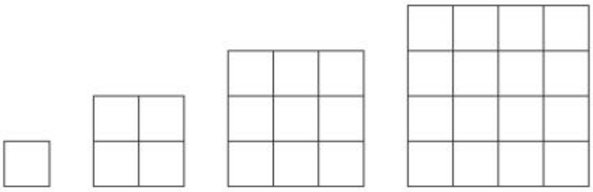
Volume of a cylinder is $V = \pi r^2 h$

$$S = 2\pi \times 1.38^2 + 2\pi \times 1.38 \times 8$$
$$S = 81.5 \text{ cm}^2$$

$S = 2\pi r^2 + 2\pi rh$

Investigation 7.1

1 The diagram shows a sequence of shapes made from cm squares.



Copy

side	1 cm	2 cm	3 cm	4 cm	5 cm	6 cm	7 cm	8 cm	9 cm	10 cm
area		4 cm^2								

Complete the table.

What are the areas multiplied by when the lengths are multiplied by 2?

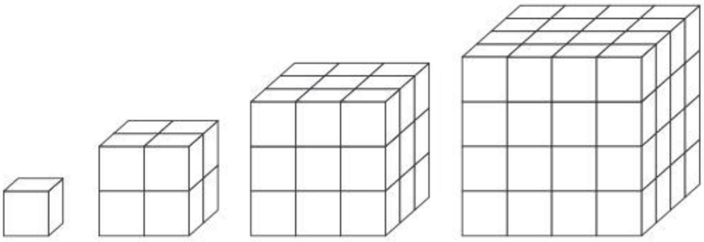
What are the areas multiplied by when the lengths are multiplied by 3?

Continue your investigation to find multipliers when you multiply by 4, 5, etc.

Generalize your result to find the multiplier when you multiply the side by n.

The multiplier when you increase the side from 2 to 3 is $\frac{3}{2}$. Verify that your result applies to rational multipliers as well as integer multipliers.

2 The diagram shows a sequence of shapes made from cm cubes.



Copy and complete this table:

side	1 cm	2 cm	3 cm	4 cm	5 cm	6 cm	7 cm	8 cm	9 cm	10 cm
volume		8 cm^3								

Complete the table.

What are the multipliers of the volumes when you multiply the sides by 2, 3, 4, etc?

Generalize your result to find the multiplier when you multiply the side by n.

Verify that your result applies to rational multipliers as well as integer multipliers.

3 Cardano's Pizzeria sells pizzas that are 20 cm, 28 cm and 35 cm in diameter. (All the pizzas have the same thickness).

a How much bigger is the 28 cm pizza than the 20 cm pizza?

b How much bigger is the 35 cm pizza than the 20 cm pizza?

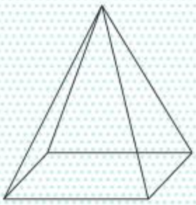
DP ready **MAA & MAI**

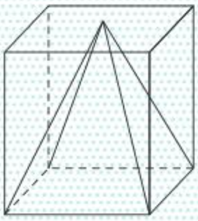
This investigation is suitable for students intending to take either MAA or MAI at HL.

Higher Level

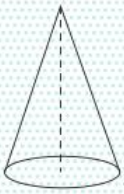
Volume of pyramid, cone and sphere

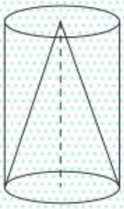
If you take a rectangular based pyramid and draw a cuboid of the same height on top of the base of the pyramid, you will find that the volume of the pyramid is one-third the volume of the cuboid.





Similarly, if you take a cone and draw a cylinder of the same height on the base, the volume of the cone is one-third of the volume of the cylinder.





DP link

You need to use a branch of Calculus to understand why this is the volume of a sphere. You will meet calculus in either the MAI or the MAA courses.

Key point

The volume of a square-based pyramid with base length l, base width w and height h is $V = \frac{1}{3}lwh$

Key point

The volume of any pyramid is $V = \frac{1}{3} \times (\text{base area}) \times h$

Key point

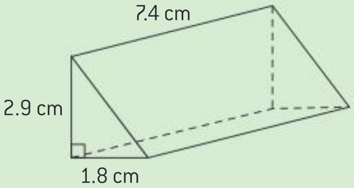
The volume of a cone with base radius r and height h is $V = \frac{1}{3}\pi r^2 h$

Key point

The volume of a sphere is $V = \frac{4}{3}\pi r^3$

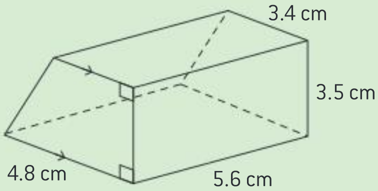
Exercise 7.3b

1 A prism has a cross-section that is a right-angled triangle with base 1.8 cm and height 2.9 cm. The length of the prism is 7.4 cm. Calculate its volume and its surface area.



2 An open-topped box is a prism with a hexagonal base. The box is to be painted inside and out. If the area of the base is 10.4 cm^2, the sides of the base are each 2 cm and the height of the box is 12 cm, calculate the area to be painted.

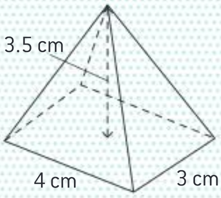
3 The ends of a prism are trapezoids. The dimensions are as shown in the diagram.



Calculate the volume and the surface area of the prism.

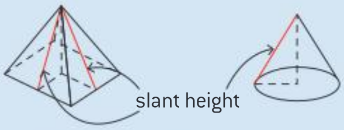
Higher Level

4



Note

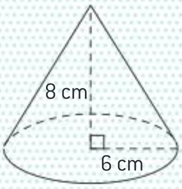
The slant height of a pyramid or cone is the distance up the sloping face.



Calculate the volume of the rectangular-based right pyramid.

Calculate the surface area of the pyramid. You will need to find the slant-height of the triangular faces using Pythagoras' theorem.

5

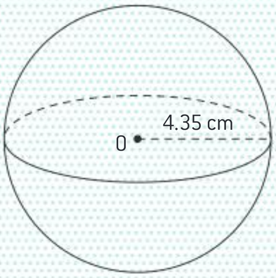


Calculate the volume of the right circular cone.

The surface area is given by the formula A = base area + $\pi \times$ radius $\times$ slant height.

Calculate the surface area of the cone.

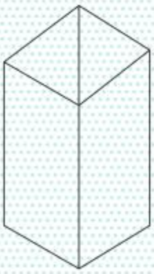
6



A sphere has a radius of 4.35 cm. Calculate its volume.

The surface area of a sphere is given by the formula $A = 4\pi r^2$. Calculate the surface area.

7 One Canada Square is a tall building in London. The shape of the building is a cuboid with a roof in the shape of a pyramid.

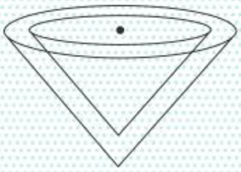


Hint

You will need to find the height of the cuboid, calculate the volumes of the cuboid and pyramid separately and add these volumes together.

The overall height of the building is 235 m and the pyramid is 40 m high. The building and the pyramid are 30 m square. Calculate the volume of the building.

8 A glass container is made so that both the outer shape and the space in the middle are cones.



The outer cone has height 7.5 cm and the inner cone has height 6.0 cm. The radii of the cones are 4 cm and 3.2 cm respectively. Calculate the volume of glass used to make the container.

Hint

Calculate the volumes of both cones and subtract the inner volume.

DP style **Applications and Interpretation HL**

9 A gold ingot is in the shape of a trapezoidal prism with parallel sides of length 178 mm and 172 mm, and a height of 44 mm. It has a width of 92 mm.

a Find the volume of the ingot.

The density of gold is 19.3 g/cm^3

b Find the weight of a gold ingot.

The gold ingot is weighed and found to only be 11.5 kg in weight. When checked further it is found to be a mixture of gold and silver rather than pure gold.

The density of silver is 10.5 g/cm^3.

c Let the volume of silver be s and the volume of gold be g.

i Write down two equations using s and g.

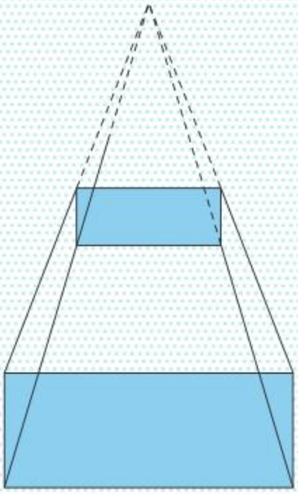
ii Hence find the volume of silver in the ingot.

The price of gold is \$41.68 per gram and the price of silver is \$0.49 per gram.

d Find the percentage reduction in value due to the gold being replaced by silver, giving your answer to the nearest percent.

DP style **Analysis and Approaches HL**

10 A pyramid with a square base of length a has a volume V and a surface area A.



The smaller pyramid, whose height is half that of the whole pyramid, is removed to leave a shape known as a frustum, as shown in the diagram.

a Find, in terms of V the volume of the small pyramid and hence the volume of the frustum.

b Find the surface area of the frustum in terms of A and a

Chapter summary

☐ The **perimeter** of a two-dimensional figure is the distance around its boundary. Perimeter is a measure of length so it is measured in cm, m, or km etc.

☐ The **area** of a two-dimensional figure is a measurement of the amount of space inside its boundary. Area is measured in square units, such as cm^2.

☐ The perimeter of a rectangle is $2(l+w)$

☐ The area of a rectangle is lw

☐ The area of the parallelogram is bh

☐ The area of the triangle is $\frac{1}{2}bh$

☐ The area of a trapezoid is $\frac{1}{2}(a+b)h$

☐ The ratio of a circle's circumference (C) to its diameter (d) is π. $\frac{C}{d}=\pi$

☐ Circumference of a circle $C=2\pi r$

☐ Area of a circle $A=\pi r^2$

☐ Volume is the amount of three-dimensional space something takes

☐ Surface area is the total area of the surface of a three-dimensional object

☐ Volume of a cuboid, $V=lwh$

☐ Surface area of a cuboid, $S=2lw+2wh+2hl$

☐ The volume of a prism, $V=Ah$

☐ The volume of a cylinder, $V=\pi r^2h$

☐ The surface area of a cylinder, $S=2\pi r^2+2\pi rh$

☐ The volume of any pyramid is $V=\frac{1}{3}\times(\text{base area})\times h$

☐ The volume of a cone with base radius r and height h is $V=\frac{1}{3}\pi r^2h$

☐ The volume of a sphere is $V=\frac{4}{3}\pi r^3$

Chapter 7 test

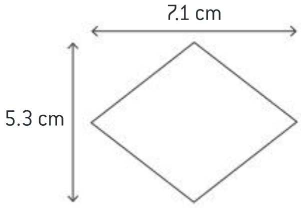
1 A circle has a radius of 3.8 cm. Find its circumference and area.

2 A circle has a diameter of 9.8 cm. Find its circumference and area.

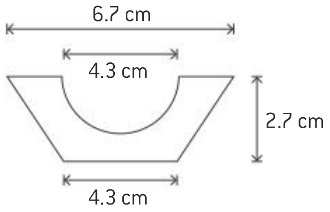
3 Find the radius of a circle with a circumference of 23.2 cm.

4 Find the diameter of a circle with an area of 12.7 cm^2

5 Find the perimeter and area of this rhombus.

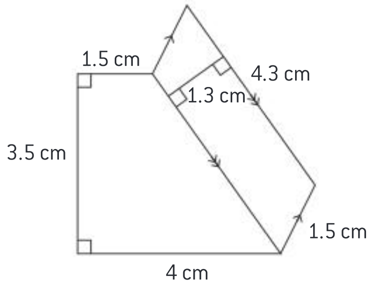


6 A shape is made by cutting a semicircle from a trapezoid.

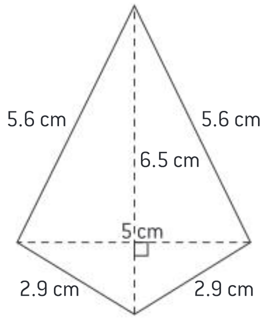


Calculate the perimeter and area of the shape.

7 Find the length of the perimeter and the area of this compound shape.



8 Find the length of the perimeter and the area of this kite.



9 An open box is in the shape of a cuboid. The box measures 5.2 cm by 15.2 cm by 7.1 cm. The open side is the largest face. Calculate the capacity of the box in ml and the surface area (excluding the open side).

10 A chocolate bar is shaped as a triangular prism. The bar is 8.2 cm long and the triangle is equilateral with a side of 2.3 cm.

a Calculate the volume of the bar.

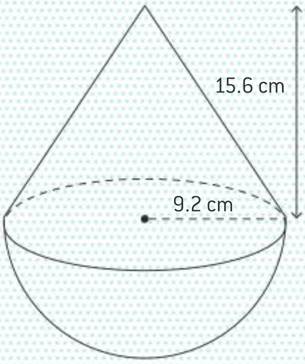
b A box is made for the bar. What area of cardboard is needed to cover the bar?

Higher Level

11 The manufacturers reduce the width of a cuboid-shaped chocolate bar by 10% and thickness by 5%, keeping the length the same. What is the overall change to the volume of the bar?

12 A hemisphere is half a sphere. The volume of a hemisphere with radius r is $V = \frac{2}{3}\pi r^3$

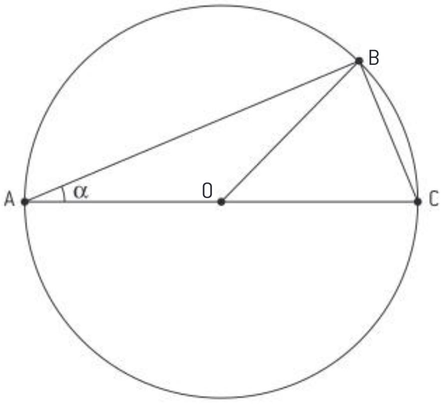
The volume of a cone with height h and radius r is $V = \frac{1}{3}\pi r^2 h$



A child's toy is made from a cone with base radius of 9.2 cm and height 15.6 cm attached to a hemisphere that also has a radius of 9.2 cm. Calculate the volume of the toy.

DP style **Analysis and Approaches SL**

13 The diagram shows a circle with centre O, and angle $B\hat{A}C = \alpha$

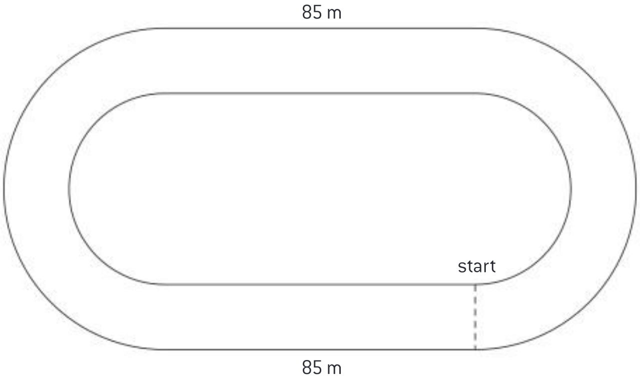


a For $\alpha = 25°$ find the size of angle $O\hat{B}C$

b Prove $A\hat{B}C = 90°$ for all values of α such that $0 < \alpha < 90°$

DP style **Applications and interpretation SL**

14 An athletics track consists of two straights of length 85 m joined by two semicircles. The total distance around the inside lane of the track is 400 m.



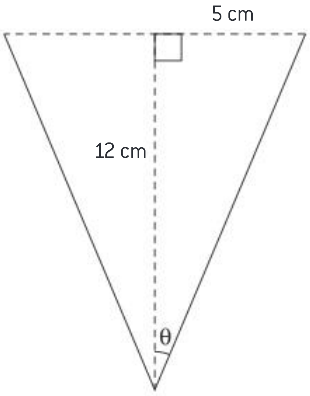
a Find the straight line distance between the inside lanes of the two straights.

The track consists of 6 lanes and the width of each lane is 1.2 m. In a 400 m race the person running in the inside lane starts the race at the end of one of the straights as shown.

b Find how far from the start line the runner in the outside lane has to be to ensure he runs 400 m.

DP style **Analysis and Approaches SL**

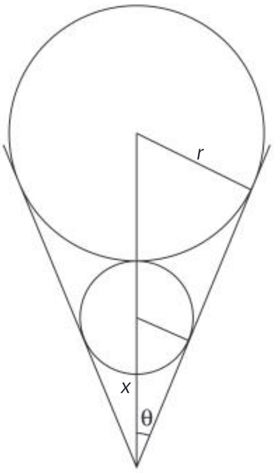
15 The cross-section through the centre of an ice cream waffle cone forms a triangle as shown. The radius of the top of the cone is 5 cm and the height is 12 cm.



a Find the surface area of the outside of the cone.

b Write down the value of $\sin\theta$

The owner of an ice cream shop is very particular about the appearance of the cones of ice cream sold in his shop. He first of all adds a sphere of ice cream with radius 2 cm and then a second larger sphere that just touches the first and also the sides of the cone. This is shown in the diagram below.



Let the distance from the apex of the cone to the centre of the first sphere of ice cream be x cm and the radius of the large sphere be r.

c Write down two ratios equivalent to the value of $\sin\theta$.

d Hence find:

- **i** the value of r
- **ii** the volume of the large sphere of ice cream.

e Determine the height of the ice cream above the top of the cone.

Modelling and investigation

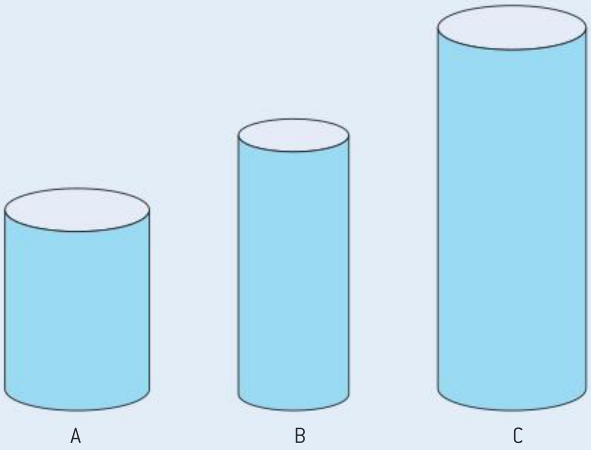
DP ready **Approaches to learning**

Critical thinking: Analyzing and evaluating issues and ideas

Organization skills: Managing time and tasks effectively

As you will learn during the Theory of Knowledge course in the IB Diploma, Mathematics is a powerful method for confirming or countering our initial thoughts or our intuition.

For example, consider the diagrams below, which show three drinks glasses.



1 Before doing any calculations, write down which of the glasses you think has a height greater than the circumference of its circular cross-section.

2 The heights and diameter of the glasses are given below. Calculate the circumference for each glass. Did your calculations confirm your intuition?

A: Height 9.5 cm, diameter 8 cm

B: Height 14 cm, diameter 6 cm

C: Height 20 cm, diameter 8 cm

At other times, intuition can be a useful way of supporting deductions in mathematics.

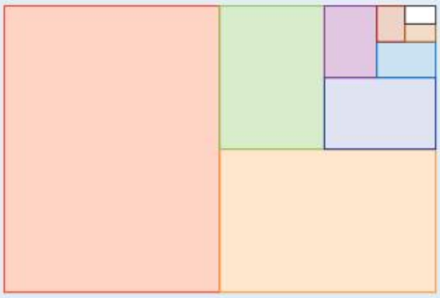
3 Consider whether it is possible to add an infinite number of positive numbers and still have a finite total.

What property would this sequence of numbers have to have?

Your initial thought may have been that this is impossible. On this occasion though intuition can help show it is.

Consider the series of numbers $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ in which each successive number is half the previous number.

4 **a** The diagram below is of a piece of coloured paper with an area of 1 unit squared. What is the link between the series given above and this diagram?

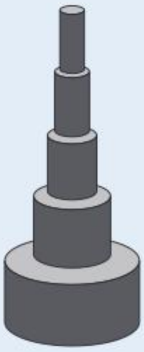


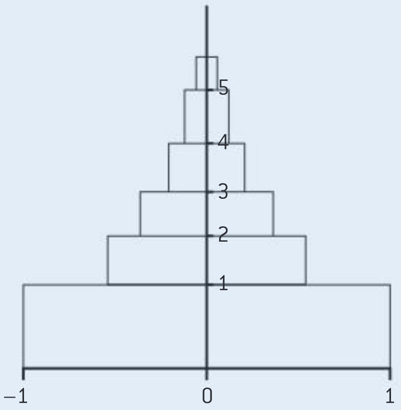
b Use the diagram to explain why $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots<1$

Though some mathematical statements, such as the sum of an infinite series of positive numbers being finite, at first seem counter intuitive, a simple illustration is sometimes enough to show it can be true.

There are occasions though where the mathematics leads to results that remain counter-intuitive even when shown to be true, and at these times you need to decide how to reconcile the mathematics and your intuition. An example of this is Gabriel's wedding cake.

Gabriel's wedding cake consists of a series of cylinders each of height 1 cm. The radius of the first is 1, the second is $\frac{1}{2}$, the third $\frac{1}{3}$ and so on.





The volume of the wedding cake (V) can be found by summing the series

$$V = a \quad 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

5 **a** Find the value of a.

b Explain why

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\ldots<1+\frac{1}{2} \quad \frac{1}{2}+\frac{1}{2} \quad +\frac{1}{4} \quad \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} \quad +\ldots$$

c Use the answers to 4b, 5a and 5b to show that $V < 2\pi$ cm^3

This result shows that though the wedding cake has infinite height the volume is actually quite small.

We will now consider how much icing is required to cover the cake.

The area of the curved sides of the cake (A) can be found by summing the series

$$A = b \quad 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

6 **a** Find the value of b.

b Explain why $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots>1+\frac{1}{2}+ \quad \frac{1}{4}+\frac{1}{4} \quad + \quad \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} \quad +\ldots$

c Hence explain why the curved area of the cake will be greater than any finite number.

Now consider what has just been proved: Gabriel's wedding cake has a finite (and quite small) volume, but it would require an infinite amount of icing to ice it.

This certainly is a counter-intuitive result!

8

Statistics

Learning outcomes

In this chapter you will learn about:

- The collection of data and its representation in bar charts, pie charts, pictograms, and line graphs
- Obtaining simple statistics from discrete data, including mean, median, mode, range

Key terms

- Data
- Frequency table
- Pictogram
- Bar chart
- Line graph
- Population
- Sample
- Random sample
- Pie chart
- Statistic
- Central tendency
- Mean
- Median
- Mode
- Spread
- Range

8.1 Collection and representation of data

DP ready **International-mindedness**

The United Nations Statistical Commission was established in 1947. Its aim is to review economic and social progress, particularly in the developing countries in the world. The goal is to help improve the lives of people around the world. Statistics help us to see where improvements need to be made in society, and show where progress has been made.

Key point

Data is a collection of facts, such as numbers, words, measurements, observations or descriptions of things.

Key point

A **frequency table** is a table listing the number of times that something occurs.

Key point

A **pictogram** is a graph that uses pictures or symbols to show the value of the data.

Example 1

Francisca's class has 20 students. Her teacher has told the class that they have to collect some data to illustrate some information about their friends. She collects data about their hair colour. Francisca's responses to the following questions are shown in the worked example.

a Design a questionnaire to collect the data.

b List the data found.

c Make a frequency table for the data.

d Draw a pictogram.

a

What colour is your hair?
(choose one)

black ☐
brown ☐
blonde ☐
red ☐

When she writes the questionnaire, she gives only four options to the students: black, brown, blonde and red so that she does not get too many different responses.

b

black, brown, brown, blonde, black, black, blonde, red, black, black, brown, blonde, black, brown, black, brown, brown, black, black, black.

She simply lists the responses in no particular order. This is not a clear way to represent the data. A table, shown in part **c**, is much clearer.

c

Hair colour	Frequency
black	10
brown	6
blonde	3
red	1

To make the frequency table, she counts the number of responses to each of the four categories.

d

black	☺☺☺☺☺
brown	☺☺☺
blonde	☺◖
red	◖

To draw a pictogram, she uses the ☺ symbol to stand for 2 people. Half of this symbol, ◖ stands for 1 person.

Key point

A **bar chart** is a graph with rectangular bars to show the frequency of the data.

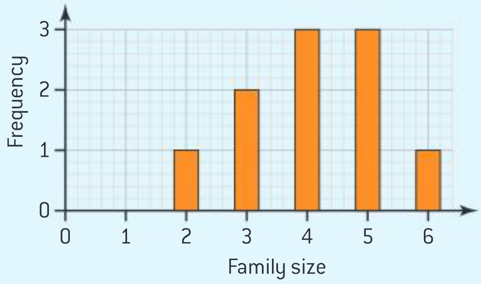
Example 2

This frequency table shows the sizes of a group of students' families.

Family size	Frequency
1	0
2	1
3	2
4	3
5	3
6	1

Draw a bar chart to show this data.

Bar chart to show the sizes of students' families



In a bar chart, frequency is shown on the vertical axis and the bars are individually labelled on the horizontal axis.

Sometimes, when there are lots of individual pieces of data, it is easier to first **group** the data before you plot the bar chart. We sometimes say that we split the data into **intervals**. Then, when you plot the bar chart, you can see patterns in the data more clearly. Example 3 shows an example of this.

Example 3

Class IB1 takes a mathematics test and their marks are recorded in the list below:

58 56 77 65 63 82 59 71 64 75 73 69 82 70 59 55 60 61 73 66.

a From this list, construct a grouped frequency table using the groups 55 – 59, 60 – 64, etc.

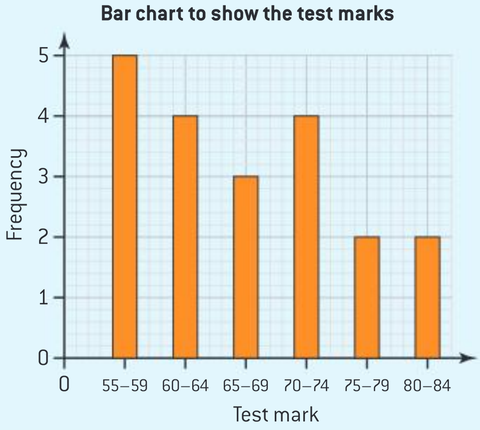
b Use your table from part **a** to draw a bar chart of the grouped data.

a

mark	frequency
55 – 59	5
60 – 64	4
65 – 69	3
70 – 74	4
75 – 79	2
80 – 84	2

The groups are of equal sizes and cover the range of marks from lowest to highest. Take care that there is no overlap, so that marks do not fall into more than one category.

b



In a bar chart, frequency is shown on the vertical axis and the bars are individually labelled on the horizontal axis with the groups of marks.

The bars have a space between them.

In Example 3, the test marks could only be whole number values in each interval. For example, test scores in the interval 55 – 59 could only be either 55, 56, 57, 58 or 59.

Some data does not have to be whole number values. Most data that is *measured*, such as a person's height or weight, can take any value in an interval. They don't have to be whole numbers.

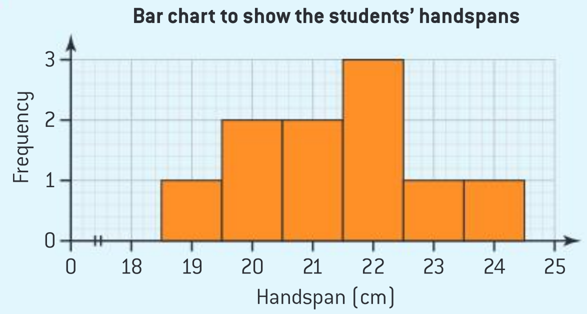
Example 4

This table records the handspan (in cm) of 10 students.

Handspan (cm)	Frequency
$18.5 < h \leq 19.5$	1
$19.5 < h \leq 20.5$	2
$20.5 < h \leq 21.5$	2
$21.5 < h \leq 22.5$	3
$22.5 < h \leq 23.5$	1
$23.5 < h \leq 24.5$	1

When measuring the handspan, values can take any value in one of the intervals.

Draw a bar chart of this data.



In a bar chart, show frequency on the vertical axis.

With this type of data, because the lengths of the handspan can be anywhere in the interval, draw the bars to show this. There are no gaps between them, because this bar chart shows a continuous range of handspan measures.

Label the axes from 18 to 25. The boundaries of each group lie in the middle, between two of these values.

Since the axis does not extend all the way to 0, use ⫲ to show that it is broken.

Line graphs are useful to show changes in a statistic over a period of time, for example. The data might be frequency or some other numerical value.

Key point

A **line graph** is a graph with points to show the data and with the points connected by straight lines.

Example 5

This table shows the population, in millions, of Beijing from 1980 to 2015.

Year	Population (millions)
1980	5.37
1985	6.02
1990	6.79
1995	8.36
2000	10.26
2005	12.99
2010	16.44
2015	18.42

www.statista.com

Draw a line graph to show this data.

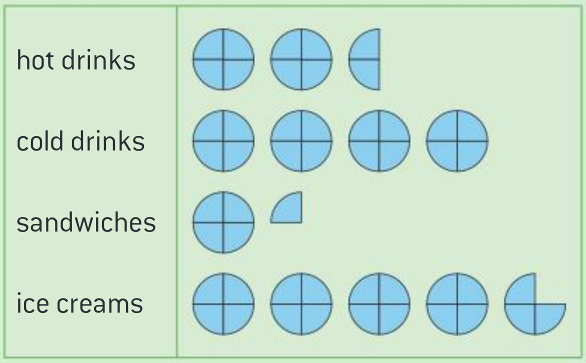
The vertical axis shows population and the horizontal axis shows the year.

Plot the value of the population for each year as a point on the graph (year, population) and then join these points with straight lines.

Exercise 8.1a

1 A café sells hot drinks, cold drinks, sandwiches and ice creams. The pictogram shows the numbers sold on Wednesday.

The ⊕ symbol in the pictogram represents 4 items sold.



a Write down how many of each item the café sold.

b Suggest what the pictogram tells us about the weather on Wednesday.

2 Students are given the choice of studying either physics, chemistry, biology or design technology. The frequency table shows the number of students choosing each subject.

Subject	No. of students
physics	7
chemistry	6
biology	12
design technology	5

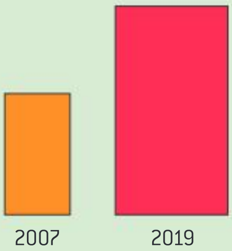
Draw a bar chart to show the number of students choosing each subject.

3 The frequency table shows the amount of rain falling each day in April.

Rainfall (mm)	Frequency (days)
$0 \le r < 5$	2
$5 \le r < 10$	3
$10 \le r < 15$	6
$15 \le r < 20$	9
$20 \le r < 25$	10

Draw a bar chart to show the rainfall in April.

4 The number of mobile phones owned worldwide has risen from 3 billion in 2007 to 5 billion in 2019.



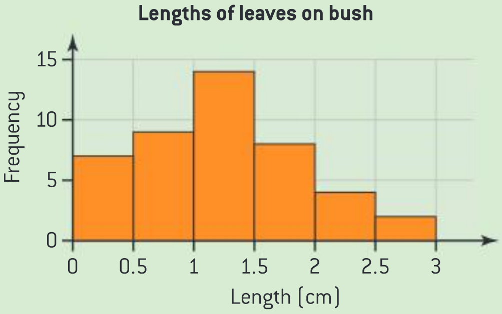
Comment on whether this diagram shows the information correctly. Give reasons for your answer.

5 This table shows the percentage of children of secondary educational age who are attending schools in some of the regions of the world.

Sub-Saharan Africa	36
Middle East and North Africa	68
South Asia	52
East Asia and Pacific	83
Latin America and Caribbean	77

Draw a bar chart to show this data.

6 This bar chart shows the lengths of leaves on a bush.



a A leaf measures 1.75 cm. Write down which length interval on the bar chart this leaf would be in.

b State how many leaves are less than 1 cm long.

c Find how many leaves were measured from the bush.

Sampling

With very large amounts of data, it is difficult to examine the whole set of data. For example, if you wanted to study the people attending a football match, rather than trying to find out about the whole crowd, you could look at a smaller number of people chosen from the crowd.

The whole crowd is called the **population** and the group you have chosen is called a **sample**. In order to make the sample representative of the whole group, it should be a **random sample**.

A sample also needs to be representative of the whole population. For example, at the football match, you would avoid gathering data from a group sitting close to each other as they are likely to be supporters of the same team. Instead you could pick random seat numbers from around the stadium and gather your data from the person sitting at each seat.

Key point

A random sample is one where every member of the population has an equal chance of being selected.

DP ready **Theory of knowledge**

Opinion polls are used to predict voting in elections and referendums. Random samples of voters are chosen and their voting preference taken. From this random sample of the voting population, predictions about the results can be made. Modern sampling techniques claim to produce results that are very accurate.

In the US presidential election in 1948, all the polls taken had predicted that Dewey would defeat Truman. In the event, however, it was Truman who won, much to everyone's surprise. In the 2016 polls, although Clinton won more votes than Trump overall, due to the American system of choosing their president, it was Trump who won the election. Polls did predict the voting correctly but they failed to forecast the right successful candidate.

Investigation 8.1

Each year a school district tests all 11-year old students in mathematics and their test results are recorded as a percentage. The following tables show random samples of 100 students taken from the results for 2017 and 2018.

2017 results

43	68	71	87	59	34	60	73	87	62
70	38	81	46	47	53	93	47	65	97
25	34	41	61	52	49	32	61	55	66
54	63	80	77	52	49	45	89	65	53
65	58	36	90	49	49	56	44	49	5
53	43	23	31	39	65	47	75	49	61
68	64	67	36	60	41	80	33	20	71
60	45	44	50	27	49	58	41	69	46
27	97	55	33	71	41	35	60	70	50
31	75	76	59	40	59	50	74	32	69

2018 results

54	48	34	53	29	26	57	30	50	27
56	69	45	57	56	49	35	34	35	53
35	32	71	46	53	35	43	37	44	61
25	49	48	50	51	51	24	42	44	44
30	50	28	21	46	59	55	54	51	55
51	36	55	45	44	60	45	49	55	28
72	56	42	56	40	30	20	43	38	64
70	36	61	59	60	60	25	9	33	41
36	47	51	30	38	25	81	41	56	23
43	58	67	47	48	25	61	70	62	40

1 Complete these grouped frequency tables for the two years.

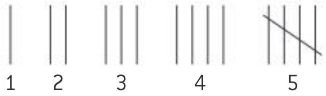
2017 results	
Mark (m)	Frequency
$0 \le m \le 9$	
$10 \le m \le 19$	
$20 \le m \le 29$	
$30 \le m \le 39$	
$40 \le m \le 49$	
$50 \le m \le 59$	
$60 \le m \le 69$	
$70 \le m \le 79$	
$80 \le m \le 89$	
$90 \le m \le 100$	

2018 results	
Mark (m)	Frequency
$0 \le m \le 9$	
$10 \le m \le 19$	
$20 \le m \le 29$	
$30 \le m \le 39$	
$40 \le m \le 49$	
$50 \le m \le 59$	
$60 \le m \le 69$	
$70 \le m \le 79$	
$80 \le m \le 89$	
$90 \le m \le 100$	

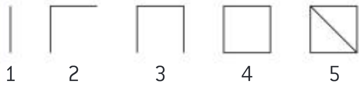
You may find the method of *tallying* useful as a way of counting the frequencies.

DP ready **International-mindedness**

In many European countries, North America, Australia and New Zealand this method is used:



France, Spain, South America and parts of Africa use this method:



In China and parts of Southeast Asia, this method is used:

These five strokes make the character 正 (zhèng), which means "correct".

When tallying, work through the list or raw data in order, making a tally mark in the appropriate group for each data value.

This is usually more accurate than counting the numbers in each group. Once the tally marks have been drawn, then the frequencies can easily be found by counting in 5s.

2 From the grouped frequency table draw two bar charts, one for each year.

3 Comment on the similarities and differences between the data based on the evidence of the bar charts.

4 Use your GDC to generate 10 random integers between 1 and 100. (Enter randInt(1, 100, 10) or RanInt#(1, 100, 10))

Use these values to choose positions in each list of 100 marks and pick the corresponding 10 data values at random. This is a random sample.

Using the same intervals, make a frequency table and bar chart for the sample.

Comment on the similarities and differences between the original sample of 100 and the sample of 10.

Pie charts

Key point

A **pie chart** is a circle divided into sectors, where each sector shows the relative size of each value.

Example 6

Francesca has collected data from her class and has made a frequency table for family size.

family size	frequency
1	0
2	1
3	2
4	3
5	3
6	1

Draw a pie chart to show this data.

$0 + 1 + 2 + 3 + 3 + 1 = 10$

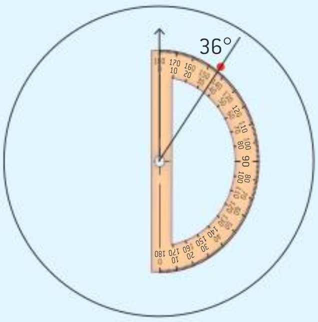
$360 \div 10 = 36°$

Add the frequencies to find the number of families.

Divide 360° by the number of families to find the angle to represent one family.

family size	frequency	Angle
1	0	$0 \times 36 = 0°$
2	1	$1 \times 36 = 36°$
3	2	$2 \times 36 = 72°$
4	3	$3 \times 36 = 108°$
5	3	$3 \times 36 = 108°$
6	1	$1 \times 36 = 36°$

Add another column to the frequency table to calculate the angles.



Draw a circle and draw a radius. Using a protractor, draw a line at 36° to the radius.

Measure a second angle, starting from the new line and so on around the circle.

Internal link

See section 5.4 for how to do this.

Either, label each sector or draw a *key* like this

🟦	2
🟩	3
🟧	4
🟨	5
🟪	6

A key would work best if the sectors were shown in different colours.

The chart should have a title.

Exercise 8.1b

1 Students are given the choice of studying either physics, chemistry, biology or design technology. The frequency table shows the number of students choosing each subject.

Subject	No. of students
physics	7
chemistry	6
biology	12
design technology	5

Calculate the angles of the sectors and draw a pie chart to show this information.

2 This table shows the age distribution of the population of China in 2010.

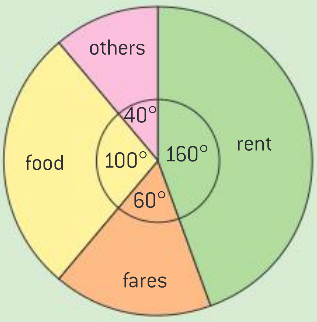
age	population (millions)
0 – 19	320
20 – 39	440
40 – 59	390
60 – 79	160
80+	20

Find the total population (in millions) and calculate the angle that will represent 1 million people.

Add a column to the table and calculate the angles of the sectors in a pie chart.

Draw a pie chart to show the age distribution of the population of China in 2010.

3 This pie chart shows Sven's weekly spending on his rent, fares, food and other items.



a What percentage of his spending is on rent?

b If he spends $75 on food, how much does he spend on fares?

c How much does he spend in total during the week?

8.2 Simple statistics

A **statistic** is a value which is calculated from some data and which can be used to represent some property of the data.

The term *average* is used in everyday language for a single number that represents a whole list of data. In mathematical terms, there are several different types of average you can find.

Different types of average all measure something known as **central tendency**. The first of these is the arithmetic mean (or **mean**) of the data.

Key point

The mean is the sum of all the values of the data divided by the number of pieces of data.

Example 7

Find the mean of

a 1, 3, 2, 5, 4, 2, 3, 1, 4, 2 **b** 12.2, 15.4, 16.6, 13.8

a $\frac{1+3+2+5+4+2+3+1+4+2}{10}=\frac{27}{10}$ $=2.7$	Count the numbers – there are 10 Add the numbers together and divide the sum by 10.
b $\frac{12.2+15.4+16.6+13.8}{4}=\frac{58}{4}$ $=14.5$	Add the numbers and divide by 4.

If a data value is missing from a list but you know the mean, you can use it to calculate the missing data value.

Example 8

In a game of darts Elizaveta scores 4, 12, 9 and 6. She throws a fifth dart and her mean score is 10. What did she score with her fifth dart?

$5\times 10=50$	The mean of five scores is 10, so total ÷ 5 = 10. Hence total = 5 × 10.
$4+12+9+6+x=50$ $31+x=50$ $x=19$	Let the fifth score be x. The sum of the five numbers is 50.

A second measure of central tendency is the **median**.

Key point

The median is the middle number when the numbers have been sorted in ascending order. (If there are two middle numbers, the median is the mean of the middle two).

2 3 3 5 7

M

4

2 3 3 | 5 7 8

M

In the example 2, 3, 3, 5, 7, there are 5 numbers and the middle number is the third, which is 3. In the second example 2, 3, 3, 5, 7, 8, there are 6 numbers. There is no middle number, the middle lies between the third and fourth. These values are 3 and 5 and so the median is $\frac{3+5}{2}=4$.

With 3 numbers the middle number is the 2nd.

5 9 10

With 4 numbers the middle number is the mean of the 2nd and 3rd. Call this the number in position $2\frac{1}{2}$.

8

6 7 | 9 11

What position would the middle number be in when there are 27 numbers or 42 numbers?

Note

Unlike the mean, the median does not take all the data values into account, it uses only one or two values at the centre of the data.

What position would the middle number be in when there are n numbers?

Example 9

Find the median of

a 2, 8, 3, 5, 1, 3, 4 **b** 12.6, 14.5, 11.9, 17.7, 16.6, 12.9

a 1, 2, 3, 3, 4, 5, 8 The median = 3	Sort the numbers in order There are 7 numbers so the median is the 4th.
b 11.9, 12.6, 12.9, 14.5, 16.6, 17.7 The median $= \frac{12.9+14.5}{2} = 13.7$	Sort the numbers in order There are 6 numbers so the median is the mean of the 3rd and 4th numbers.

Key point

The median of a set of n numbers is in position $\frac{n+1}{2}$.

Look at this set of data: 1, 2, 7, 1, 3, 13, 1, 1, 2, 1, 1. The mean is 3. Notice how most of the numbers are less than the mean.

Written in ascending order, this set becomes 1, 1, 1, 1, 1, 1, 2, 2, 3, 7, 13. So the median is the 6th number in the list which is 1. As a measure of central tendency, the median works better here than the mean.

However, to best represent the data you might think that 1 would be better. The value 1 is the **mode** of the data.

Key point

The mode is the number that appears most often in a list.

Example 10

Find the mode of:

12, 11, 13, 11, 12, 12, 11, 14, 12, 11, 13, 12, 11, 13, 12

12 is the number which occurs most often (6 times), so the mode is 12.	The list has five 11s, six 12s, three 13s and one 14.

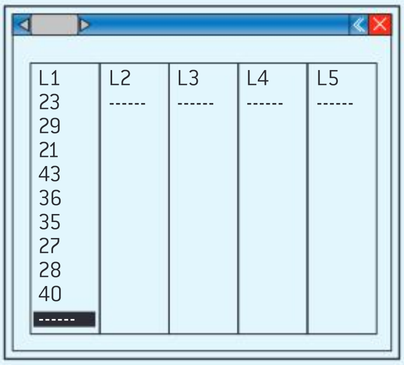
You can use your GDC to find the mean and the median using lists.

Example 11

The lengths (in mm) of the leaves on a plant are 23, 29, 21, 43, 36, 35, 27, 28, 40.

Calculate the mean and median of the data.

Enter the data in a list.

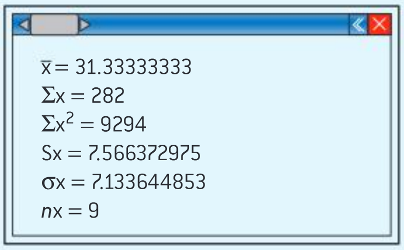


L1	L2	L3	L4	L5
23	------	------	------	------
29				
21				
43				
36				
35				
27				
28				
40				

Select 1-var statistics

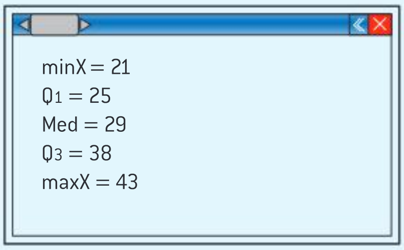
mean = $\overline{x}$ = 31.3

$$\text{mean} = \frac{\Sigma x}{n} = \frac{282}{9}$$



$\overline{x}$ = 31.33333333
Σx = 282
Σx^2 = 9294
Sx = 7.566372975
σx = 7.133644853
nx = 9

median = Med = 29



minX = 21
Q1 = 25
Med = 29
Q3 = 38
maxX = 43

Exercise 8.2a

1 Calculate the mean values of:

a 2, 5, 3, 12, 4 **b** 12.1, 17.3, 6.5, 2.4, 8.7, 9.5

2 Find the median values of:

a 4.5, 2.3, 1.9, 4.2, 5.8 **b** 7, 8, 3, 5, 8, 9, 2, 3, 4, 7, 6, 5, 2, 3

3 Rajesh is skimming stones across a lake. He counts how many bounces the stone makes before sinking and records this information:

0, 2, 4, 3, 7, 2, 1, 5, 7, 4, 3, 2, 3, 0, 2, 4, 5, 6, 3, 2

Calculate the mean, the median and the mode of the number of bounces.

4 Hussein is playing a computer game. He collects his scores. These are 20 305, 15 770, 25 320, 4120, 67 225, 61 300.

He finds the mean and median score for the game. Calculate these and hence explain whether mean or median would be a better representative of his ‘average’ score.

5 The table shows monthly sales of ice cream (in thousand dollars) for an ice cream company.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
50	60	77	96	137	158	167	159	108	75	61	54

Calculate the mean and the median monthly ice cream sales.

6 Peter records his 100 m sprint times (in seconds) during the season. They are:

15.2, 14.7, 14.3, 14.8, 13.5, 14.2, 13.4, 13.8, 12.9, 13.7

Calculate his mean and median times.

Peter realises that he has made a mistake writing down one of his times, which should have been 15.7 s and not 14.7 s.

Find the new mean and median with the corrected time.

7 The mean of six numbers is 5.2. Five of the numbers are 3.8, 5.9, 4.4, 5.1 and 5.6. Find the sixth number.

8 The mean height of 10 men is 1.7 m and the mean height of six women is 1.55 m. Find:

a the total height of the 10 men

b the total height of the 6 women

c the mean height of the whole group.

Investigation 8.2

The frequency table shows the weights of eggs bought in a supermarket.

weight (g)	58	59	60	61
frequency	2	5	8	5

1 Copy and complete this list of the data: 58, 58, 59, 59, 59, …

2 Use the list to calculate the mean and to find the median and mode for the weight of an egg from the supermarket. Check your results with your GDC.

3 There are 2 eggs that weigh 58 g, so their total weight is $58 \times 2 = 116$ g and 5 eggs that weigh 59 g so their total weight is $59 \times 5 = 295$ g. If you add all these products, you will get the total weight of all the eggs. To get the total number of eggs, you need to add the frequencies.

Continue this process to find the mean weight of the eggs bought from the supermarket and check this value is the same as you found before.

Which method is the most efficient: listing all the values or the method you have just used? Why are they equivalent?

4 How could you find the mode directly from the frequency table?

5 What was the total number of eggs? In what position is the median?

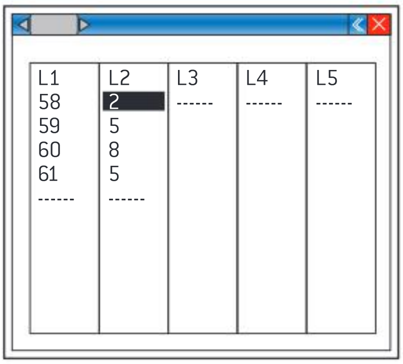
Using the frequency table, when the weights of the eggs are in order, the first 2 eggs are 58 g. The next 5 weigh 59 g. These eggs will be in position 3 to $2 + 5 = 7$. In which group will the median lie and what is the median weight?

Is this method more efficient than listing all the numbers first? Why are these methods equivalent?

6 The frequency table can be entered directly in the GDC. Work through this example to see how this is done.

Enter the data in list 1 and frequencies in list 2

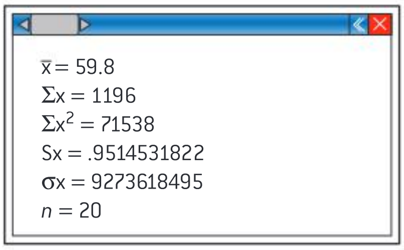
Select 1-var statistics with list 1 as the list of values and list 2 as the frequency list



```
L1      L2      L3      L4      L5
58      2       ------  ------  ------
59      5
60      8
61      5
------  ------
```

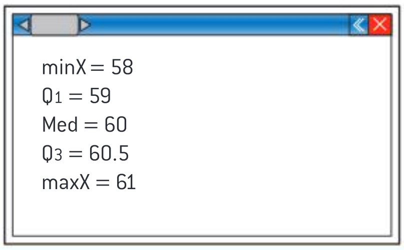
mean = $\bar{x}$ = 59.8

$$\text{mean} = \frac{\Sigma x}{n} = \frac{1196}{20} = 59.8$$



```
x̄ = 59.8
Σx = 1196
Σx² = 71538
Sx = .9514531822
σx = 9273618495
n = 20
```

median = Med = 60



```
minX = 58
Q1 = 59
Med = 60
Q3 = 60.5
maxX = 61
```

Example 12

A production line produces bags of carrots. Random samples of the carrots are taken from the line to check that they are being packed to the correct weight. The bags are labelled with a weight of 200 g.

The results are compared before and after the production line is serviced and the weights (to the nearest gram) of 50 bags collected in the samples are recorded in these frequency tables.

before	
weight (g)	frequency
198	3
199	7
200	17
201	21
202	2

after	
weight (g)	frequency
198	5
199	10
200	23
201	11
202	1

Calculate the mean weight of each sample and use it to compare the weights of the bags before and after the machinery was serviced.

before

$$\text{mean} = \frac{198 \times 3 + 199 \times 7 + 200 \times 17 + 201 \times 21 + 202 \times 2}{3 + 7 + 17 + 21 + 2}$$

$$= \frac{10012}{50}$$

$$= 200.24$$

Multiply weight × frequency and sum to get the total weight of the bags.

Divide by 50 to get the mean weight.

after

$$\text{mean} = \frac{198 \times 5 + 199 \times 10 + 200 \times 23 + 201 \times 11 + 202 \times 1}{5 + 10 + 23 + 11 + 1}$$

$$= \frac{9993}{50}$$

$$= 199.86$$

After servicing the mean weight has gone from being slightly over 200 g to being slightly under.

This suggests that the production line is now producing underweight bags. Before making any changes, however, it would be good to take more samples to check that this sample is representative.

In some situations there is more to a comparison of two sets of data than their central tendency. Another measure is the **spread** of the data. Spread can be measured using the **range** of the data.

Key point

Range is the difference between the maximum and minimum values of the numbers in the set of data.

DP link

DP students will study other measures of spread such as quartiles and standard deviation.

Example 13

Two classes take the same mathematics test. Their teachers want to compare the scores of each class.

The marks in the test are as follows:

Class 1: 67, 39, 42, 31, 34, 65, 95, 91, 68, 17, 53, 90, 27, 56, 57, 81, 30

Class 2: 67, 66, 63, 57, 50, 62, 51, 60, 42, 50, 54, 46, 63, 43, 47

Find the mean, median and range of the marks of the two classes and compare them.

mean for class 1

$$= \frac{67+39+42+31+34+65+95+91+68+17+53+90+27+56+57+81+30}{17}$$

$= 55.5$

mean for class 2

$$= \frac{67+66+63+57+50+62+51+60+42+50+54+46+63+43+47}{15}$$

$= 54.7$

Calculate the means.

median for class 1

17 27 30 31 34 39 42 53 56 57 65 67 68 81 90 91 95

median = 56

median for class 2

42 43 46 47 50 50 51 54 57 60 62 63 63 66 67

median = 54

Sort the marks into order and find the medians.

Range for class 1 = 95 – 17 = 78

Range for class 2 = 67 – 42 = 25

Subtract minimum value from maximum value to find the range.

Although both sets of data have very similar means and medians, their ranges are quite different. This shows that the marks for class 1 have a greater spread than those of class 2.

Exercise 8.2b

1 A box contains screws of different lengths. This frequency table shows its contents.

length (cm)	1	1.5	2	2.5	3
number of screws	11	14	16	19	10

Find the mean, the median and the modal length of the screws in the box.

DP style **Applications and Interpretation**

2 A bus company charges its passengers $1 per stop on the route. The number of stops that passengers travel are recorded for a day.

number of stops	1	2	3	4	5
number of passengers	19	37	54	32	14

a Calculate the total revenue for the route in one day

b The bus company decides to charge a flat fare for each journey. To keep the same total revenue for the route, they charge the mean fare paid by the passengers. How much is this?

c If they had chosen to use the median fare, instead of the mean, how would this have affected the total revenue?

3 In 2014 the mean and median household incomes in the United States were respectively $72,641 and $53,718. How would you account for the difference between the two values? What does this tell you about the distribution of wealth in the United States?

DP style **Analysis and Approaches**

4 The number of goals scored by a football team in a series of football matches is shown in this table.

number of goals	0	1	2	3
number of matches	4	5	3	x

If the mean number of goals scored is $1\frac{1}{3}$, find the value of x.

Chapter summary

☐ Data is a collection of facts, such as numbers, words, measurements, observations or descriptions of things
☐ A frequency table is a table listing the number of times that something occurs
☐ A pictogram is a graph that uses pictures or symbols to show the value of the data
☐ A bar chart is a graph with rectangular bars to show the frequency of the data
☐ A line graph is a graph with points to show the data and with the points connected by straight lines
☐ A random sample is one where every member of the population has an equal chance of being selected
☐ A pie chart is a circle divided into sectors, where each sector shows the relative size of each value
☐ The mean is the sum of all the values of the data divided by the number of pieces of data
☐ The median is the middle number when the numbers have been sorted into order. (If there are two middle numbers, you find their mean)
☐ The median of a set of n numbers is in position $\frac{n+1}{2}$
☐ The mode is the number that appears most often in a list
☐ Range is the difference between the maximum and minimum values of the numbers in the set of data

Chapter 8 test

1 Bronwyn conducts a traffic survey outside her school. In 30 minutes, she counts 45 cars, 25 trucks, 15 motor bikes, 15 bicycles and 20 buses. Using a symbol to indicate 10 of each vehicle, show this data in a pictogram.

2 Darma is a member of a sports club. The ages of the boys in the club are 14, 13, 15, 13, 12, 14, 15, 12, 13, 14, 16, 14, 14, 13, 12, 14, 12, 15, 16, 13. From this data make a frequency table and draw a bar chart to show the boys' ages.

3 Mrs Niedermeyer tells her class that they can vote for which day of the week to go on a class outing. The votes are collected in and listed:

Friday, Wednesday, Friday, Monday, Tuesday, Thursday, Friday, Monday, Tuesday, Friday, Friday, Monday

Make a frequency table of the results and use the frequencies to calculate the size of the sectors of a pie chart. Draw the pie chart.

4 The weights of 70 apples are recorded and shown in the frequency table.

Weight (g)	Frequency
$70 \le w < 75$	3
$75 \le w < 80$	7
$80 \le w < 85$	8
$85 \le w < 90$	13
$90 \le w < 95$	27
$95 \le w < 100$	9
$100 \le w < 105$	3

Show this data in a bar chart.

5 Mei Ling makes a batch of cupcakes. She weighs the cakes and they are 84 g, 72 g, 93 g, 85 g, 77 g, 78 g, 89 g. Calculate the mean weight of her cupcakes.

6 Mr Dubois asks his English class to write a fifty-word biography of themselves as an exercise. The biographies are all approximately fifty words, but he makes a list of the actual lengths. They are 51, 55, 48, 47, 53, 54, 51, 48, 49, 50, 52, 46, 48, 53, 52, 50. Calculate the median length (in words) of the biographies.

7 Mr Dean's mathematics class count the number of coins that they each have. The results are listed:

0, 2, 4, 16, 12, 8, 0, 2, 9, 4, 3, 0, 2, 1, 0, 4, 12, 11.

a Write down the mode of the number of coins.

b Calculate the range in number of coins.

8 The heights of tomato plants were measured. They were 124 cm, 205 cm, 161 cm, 207 cm, 134 cm, 245 cm, 118 cm. A new plant was measured and the mean height is now 171 cm. Find the height of the new plant.

9 The number of hours that Mr Singh's central heating was used per day is recorded over a two-week period. The data is shown in this frequency table:

Number of hours	Frequency
0	3
1	2
2	1
3	5
4	1
5	2

a Find the median number of hours per day that Mr Singh used his heating in this period.

b Find the mean number of hours per day that Mr Singh used his heating in this period.

DP style **Applications and interpretation**

10 In a football competition team A plays 10 games and team B plays 5 games. The mean number of goals scored by team A is 2.2 and the mean number of goals scored by team B is 1.8.

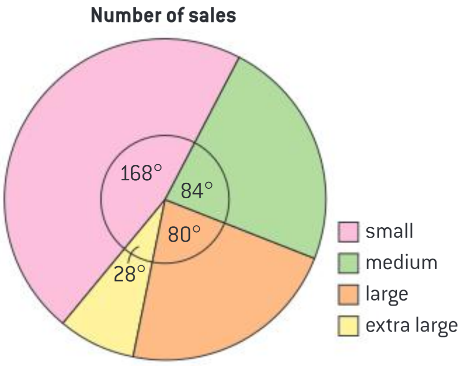
John claims that as the average of 2.2 and 1.8 is 2.0, then 2.0 must be the mean number of goals over the whole 15 games.

a John is wrong in his claim. Determine whether the mean number of goals over the whole 15 games is greater than or less than 2.0, justifying your answer.

b Team A are due to play team B in the next game. Suggest what extra information might you need before deciding which team is most likely to win.

DP style **Analysis and Approaches**

11 A manager records the sizes of the dresses sold in his shop one week and shows this information in a pie chart.



a **i** Write down the modal size of dress sold that week.

The dresses are classed as: small = size 1, medium = size 2, large = size 3, extra large = size 4.

ii Find the size category in which the median dress would sit.

b Express the number of dresses sold which were small as a percentage of the total dresses sold that week.

The following week, the manager reviews the pie chart and realizes he cannot find the original data. He remembers that there were seven extra large dresses sold.

c Find the numbers of dresses sold in each of the other categories and the total number sold that week.

The manager calculates that if he had sold just four more small dresses and no more of the other sizes then the median as well as the mode would lie in the 'small' category.

d Explain why the manager is wrong, and find the fewest number of extra small dresses that would need to have been sold for the median dress size to be 'small'.

Modelling and investigation

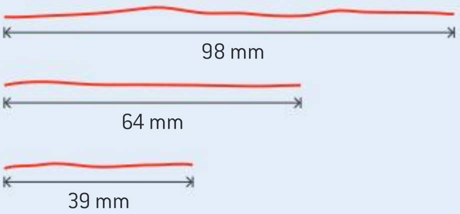
DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Organization skills: Managing time and tasks effectively

There is a theory that people *under* 20 years old are better at estimating length than people *over* 20 years old. You are going to test this theory and see whether it has any validity.

In this task, you are going to get some members of your class, friends or family to draw three straight lines. These lines should be drawn freehand on a plain sheet of paper, without a ruler. Choose the target lengths for them to draw. You must then measure the lines to see how accurate their drawings are.



Planning.

- Design a form to collect your data.
- How many people will you get to take the test so that you have enough data?
- What will the target lengths be that you get your subjects to draw?
- What will you record for each measurement? Length, difference from target? Are you concerned whether the lines are longer or shorter than the target length?
- How will you show your data in a table? If you use a grouped frequency table, what will the groups be?
- How will you display your data? Which charts are the most appropriate?
- What statistics will you calculate?
- How will you compare the data for those under 20 and those age 20 or over? What measures of average and spread will you use?

Once you have collected your data, tabulate what you have found, display the data in charts and calculate statistics.

From your data, draw a conclusion about the theory you are testing. Evaluate your findings.

9

Probability

Learning outcomes

In this chapter you will learn about:

- Calculating probabilities of simple events
- Venn diagrams for sorting data
- Tree diagrams

Key terms

- Event
- Experiment
- Relative frequency
- Trial
- Sample space
- Sample space diagram
- Contingency table
- Tree diagram
- Exhaustive

9.1 Calculating probabilities of simple events

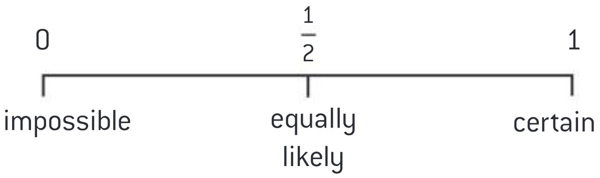
Probability is a study of how likely it is that something happens.

An **event** is something that happens when an **experiment** is performed. In probability an experiment could be something like tossing a coin, where an event would be obtaining a head, for example.

Note

Probability is now an important part of statistics, economics, science and computing.

Probability assigns a number between 0 and 1 to an event as a measure of how likely that event is. This can be expressed as a fraction, a decimal or a percentage.



Key point

The probability of an event happening = $\frac{\text{number of ways it can happen}}{\text{total number of outcomes}}$

In the example of tossing a coin, there are two equally likely outcomes (if the coin is fair):

“heads” or “tails”.

The probability of either event is $\frac{1}{2}$ or 0.5.

Another example of a probability experiment is rolling a dice,  which is a cube with its faces numbered from 1 to 6. Each of the six numbers are equally likely to be rolled, so the probability of each number is $\frac{1}{6}$.

There are two types of probability: **theoretical probability** and **experimental probability**. Theoretical probability is what you expect to happen, and experimental probability is what actually happens. Experimental probability is also called **relative frequency**.

Key point

Relative frequency of an event = $\frac{\text{number of times an event occurs}}{\text{total number of trials}}$

Key point

A **trial** is an individual repetition of an experiment.

Investigation 9.1

1 Toss a coin 20 times and record how many heads and how many tails you get. Calculate the relative frequency of getting a head.

In your class, combine your results and calculate the relative frequency of getting a head for the whole class.

2 Drop a thumbtack (drawing pin)  on a flat surface. The thumbtack can either land point-up or point-down. Drop the thumbtack 20 times and record the number of times it lands point-up and the number of times it lands point-down. Calculate the relative frequency of the thumbtack landing point-down.

Combine your results to find the relative frequency for the whole class.

How good an approximation to the theoretical probability do you think your results were? How could you get a better approximation?

If you were to toss the coin enough times, you would expect the relative frequency to be close to 0.5. Probability is however about what *should* happen and not about what *will* happen. Even though you would expect to get about 10 heads out of 20 coin tosses, sometimes you might get 5 or 15. Each trial is independent of the one before. Just because you have tossed a head in one trial, there is no more (or less) chance of getting a head or tail in the next. The law of large numbers states that the more trials are performed, the closer the relative frequency gets to the theoretical probability.

Try this coin toss simulator to investigate this further: https://www.basic-mathematics.com/coin-toss-probability.html

If A is the set of outcomes of an event then n(A) is the number of outcomes that make the event happen. The set of possible outcomes is U, n(U) is the number of possible outcomes, and U is known as the **sample space**. The probability of A is written P(A).

Key point

$$P(A)=\frac{n(A)}{n(U)}$$

Whenever you calculate the probability of an event happening, there is always a complementary event, that is the event *not* happening.

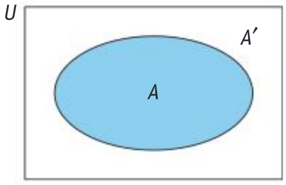
For example if the probability I win a race is 0.2 then the complementary event is that I do not win, which has a probability of 0.8.

If P(A) is the probability of event A happening, then P(A') is the probability of A *not* happening. Since either A happens or it does not, then their combined probabilities must be 1, the probability of something that is certain, and so $P(A) + P(A') = 1$. This leads to an important result:

Key point

$$P(A') = 1 - P(A)$$

You can use a Venn diagram to show the sets A, A' and U.



Internal link

See section 1.2

Example 1

a A box contains 5 red pens and 3 blue pens. Find the probability of picking out a red pen from the box.

b The probability a traffic light is red is 0.4. Calculate the probability it is not red.

a P(red pen) $= \frac{5}{8}$	The total number of pens is 5 + 3 = 8
b P(not red) $= 1 - 0.4 = 0.6$	P(not red) = 1 – P(red)

Note

In some situations, probabilities are expressed in a different way. For example, a "fifty-fifty" chance or odds of 5 to 1. In these examples the ratio is number of ways an event can happen to the number of ways it cannot. That is $\mathrm{n}(A) : \mathrm{n}(A')$.

50 : 50 is the same as 1 : 1 or $\frac{1}{2}$

Likewise, 5 : 1 is the same as $\frac{5}{6}$.

However fractions and decimals are the only acceptable ways of showing probabilities in mathematics. (Percentages could also be used).

Probability of event A or event B

A spinner is divided into seven sections all with an equal chance of coming up when the spinner is spun. Four of the sections are yellow, two green and one blue. Let Y, G and B be the events of the spinner landing on yellow, green and blue respectively.

$$\mathrm{P}(Y) = \frac{4}{7},\ \mathrm{P}(G) = \frac{2}{7} \text{ and } \mathrm{P}(B) = \frac{1}{7}$$

The probability of either yellow or green coming up $(Y \cup G)$ will be the number of ways this can happen divided by the total number of possible outcomes. Hence

$$\mathrm{P}(Y \cup G) = \frac{6}{7}$$

It can be seen that this is the same as P(Y) + P(G)

This relationship will be true whenever the two events cannot occur together. In which case we refer to the events as being **mutually exclusive**.

Key point

$\mathrm{P}(A \text{ or } B) = \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ whenever A and B are mutually exclusive.

When the two events are not mutually exclusive the probability needs to be worked out by listing all the possibilities and dividing by the total number of possibilities. Note that when a question refers to '*A* **or** *B*' it includes the possibility of both *A* **and** *B*.

Example 2

When I roll a dice let A be the event of obtaining a number less than 4 and B be the event of obtaining an odd number. Find the probability of obtaining a number less than 4 or an odd number on a single roll.

The possibilities are 1,2,3,4,5,6 Of these the numbers that are either odd or less than 4 (or both) are 1,2,3,5 hence the answer is $\frac{4}{6} = \frac{2}{3}$	The two events are not mutually exclusive as you can get a number that is both odd and less than 4. This means the answer is not $\frac{3}{6} + \frac{3}{6} = 1$ Note that 1 and 3 are included, as numbers which are both less than 4 and odd.

Exercise 9.1

1 A bag contains 4 apples and 6 oranges. A piece of fruit is taken out of the bag at random. Find the probability that it is an apple.

2 In a prize draw, a box contains 50 tickets which either say “prize” or “lose”. There are 5 tickets that say “prize”. Find the probability of winning the draw.

3 Calculate the probably that a day of the week, chosen at random, contains the letter:

- **a** “T”
- **b** “Y”

4 4 red balls, 3 blue balls and 5 yellow balls are in a bag. One ball is taken from the bag at random. Find the probability that the ball chosen is:

- **a** red
- **b** blue
- **c** red or blue.

5 Find the probability of rolling a 7 with a standard dice.

6 Find the probability of rolling a prime number with a standard dice.

7 The weather is classified as either hot, warm or cool. The probability the weather is hot is 0.6 and the probability it is warm is 0.32.

- **a** Find the probability the weather is:
  i cool **ii** cool or warm **iii** not hot.

The probability it will rain on any day is 0.45.

- **b** Explain why you cannot work out the probability that the weather will be either warm or rainy.

8 On our street there are 15 houses on the left hand side, 7 of which have green curtains. There are also 12 houses on the right hand side and of these, 6 have green curtains. A house is chosen at random. Find the probability it is either on the right hand side or has green curtains (or both these things).

DP ready **Theory of knowledge**

In the real world, probability is not about calculating how likely the outcomes of simple events like the outcomes of rolling a dice or tossing a coin are. An important area where probability is used is in decision making. For example, when a drug company introduces a new drug, two things need to be evaluated. First the effectiveness of the drug in curing the disease it is aimed at and second the likelihood of undesirable side effects. No drug is likely to be 100% effective and equally none is likely to be without unwanted side effects. Both considerations need to be evaluated along with ways of improving the drug’s efficiency and reducing risk. When this has been done, mathematical, statistical techniques are used to evaluate the relative importance of each. In the same way, weather forecasts give the percentage probability of rain or sunshine.

9.2 Sorting data

If you toss two coins, then there could be 0, 1 or 2 heads. Notice that the event of tossing 1 head can happen in one of two ways, either HT or TH.

The sample space consists of four events:

HH HT TH TT

It can be shown using a **sample space diagram** that shows each of the four possible outcomes.

	H	T
H	HH	HT
T	TH	TT

If all the events in the sample space are equally likely then probabilities can be worked out using the formula $P(A) = \frac{n(A)}{n(U)}$

For example, the probability of getting exactly one head is $\frac{2}{4} = \frac{1}{2}$

Rolling two dice results in many more events: 36 altogether. Writing out a list of all 36 events is quite difficult, but a sample space diagram makes it easier. On the sample space diagram below, the sum of the values on the two dice are shown.

First dice

Second dice

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Sample space diagrams are a useful aid to finding probabilities.

Example 3

Two dice are rolled and their scores added. Find the probabilities of obtaining:

a a “double” (two numbers the same)

b a total of less than 6.

a

First dice

Second dice

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Draw a sample space diagram. On the diagram indicate the events which are doubles.

$$\text{P(double)} = \frac{6}{36} = \frac{1}{6}$$

b

First dice

Second dice

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Draw a sample space diagram. On the diagram indicate the events where the total is less than 6. (2, 3, 4 or 5).

$$\text{P (less than 6)} = \frac{10}{36} = \frac{5}{18}$$

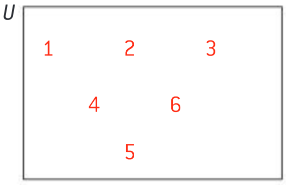
Internal link

See section 1.2 for an introduction to Venn diagrams.

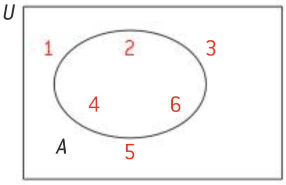
Venn diagrams for sorting data

A Venn diagram is commonly used to represent probabilities in an organised way.

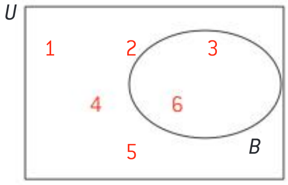
The sample space is shown by the rectangle that represents the universal set, U. For example, if a dice is rolled then $U = \{1, 2, 3, 4, 5, 6\}$ and $n(U) = 6$.



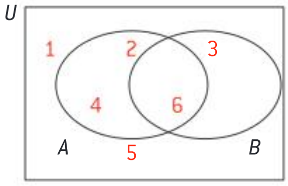
Now consider the event A “an even number”, then $A = \{2, 4, 6\}$. These outcomes can be shown in a ring labelled A. The set $A' = \{1, 3, 5\}$.



If B is the event "a multiple of 3", then $B = \{3, 6\}$. These outcomes can be shown in a ring labelled B. The set $B' = \{1, 2, 4, 5\}$.

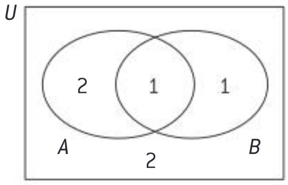


Showing both sets in the diagram with intersecting rings, you can see that $A \cap B = \{6\}$, $A \cup B = \{2,3,4,6\}$, $A \cap B' = \{2,4\}$, $A' \cap B = \{3\}$, $(A \cup B)' = \{1,5\}$, etc.



When using a Venn diagram to calculate probabilities, it is usually easier to show the number of outcomes in each ring, rather than its contents.

For example, there is one number in the intersection of A and B (that is, $n(A \cap B) = 1$) so we would put a 1 in this position in the Venn diagram. Similarly there are two numbers outside both circles so we would write a 2 here. The resulting Venn diagram is



You can then use these figures to calculate probabilities. For example, the probability of a dice roll being even and a multiple of 3 is

$$P(A \cap B) = \frac{n(A \cap B)}{n(U)} = \frac{1}{6}.$$

Conditional probability

Sometimes you are given some extra information which will change the probability associated with an event. In the example above we could ask: given that the number rolled is even, what is the probability it is a multiple of 3?

Because we know it is even there are now just three possibilities and only one of these is a multiple of 3, so the answer is $\frac{1}{3}$.

Example 4

20 students are asked about some of the subjects they are studying.

12 students study economics, 8 study design technology and 6 study Mandarin.

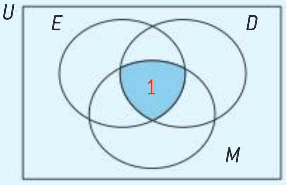
2 study economics and design technology, 3 study design technology and Mandarin, and 3 study Mandarin and economics. There is one student who studies all three subjects.

Show this information in a Venn diagram and find the probability that a student chosen at random studies:

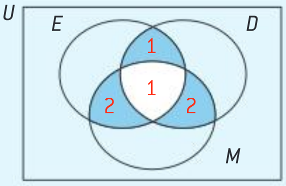
a only one of the three subjects

b exactly two out of the three subjects

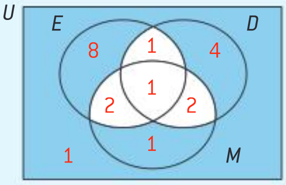
c at least 2 subjects, given they study Economics.



Enter the student who studied all three subjects in the intersection.



Enter the number of students who studied two subjects in the intersections. (Do not include the student who studied all three.)



Enter the number of students remaining in the three sets. Then, totalling the students entered so far, place any others in the complement.

a P(studies only one) $= \frac{8+4+1}{20} = 0.65$

b P(studies exactly two) $= \frac{1+2+2}{20} = 0.25$

c P(studies at least 2 given they study economics)$= \frac{1+1+2}{12} = \frac{1}{3}$

Another similar diagram that helps to organise data and helps in the calculation of probabilities is a **contingency table** or **two-way table**. In the table, there are two variables. The use of the contingency table is shown in the following table.

Example 5

The voting preferences of 100 people in a survey are shown in this table:

voting preference	blue	orange	Total
male	23	31	54
female	17	29	46
Total	40	60	100

Find the probability that a person chosen at random is:

a male

b a female voting orange

c an orange voter

d an orange voter given they are female

e female given they are an orange voter.

a $\frac{23+31}{100} = 0.54$

b $\frac{29}{100} = 0.29$

c $\frac{31+29}{100} = 0.6$

d $\frac{29}{46}$

This is now out of 46 as the total number of female voters was 46.

e $\frac{29}{60}$

This is out of 60 as the total number of orange voters was 60.

Exercise 9.2

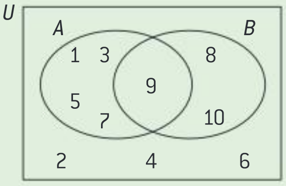
1 Find the probability of getting 2 heads when two coins are tossed.

2 Roger rolls a dice and tosses a coin. Draw a sample space to show all possible outcomes. Find the probability that he rolls an even number and tosses a tail.

3 Find the probability of scoring a total of 7 when 2 dice are rolled.

4 Calculate the probability of getting a total of 10 or more when two dice are rolled.

5 A spinner with ten equal parts numbered from 1 to 10 is spun. A is the event "the spinner lands on an odd number". B is the event "the spinner lands on a number greater than 7".



The possible outcomes of the experiment and the events A and B are shown in the Venn diagram. Use the diagram to calculate the probabilities that the spinner lands on:

- **a** a number greater than 7
- **b** an odd number greater than 7
- **c** an even number less than or equal to 7.

6 The set U consists of the numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

The numbers $\{1, 3, 4, 5, 7\}$ are in set X and $\{2, 4, 6, 7\}$ are in Y.

Draw a Venn diagram showing where all the numbers from 1 to 8 belong.

- **a** List the outcomes that are not in Y and hence write down $P(Y')$.
- **b** List the outcomes in X but not in Y and hence write down $P(X \cap Y')$.
- **c** List the outcomes in X or Y or in both, and hence write down $P(X \cup Y)$.

7 There are 15 girls in grade 10. They can play touch rugby or football. 10 girls play touch rugby and 7 play football. 4 girls play both sports. Show these figures in a Venn diagram and hence calculate the probability that:

- **a** a girl chosen at random will play neither sport
- **b** a girl chosen at random plays only touch rugby.

DP style **Applications and Interpretation SL**

8 In a class of 20 students, 10 study English, 9 study Spanish and 5 study French. None of the students study Spanish and French. 4 study English and Spanish and 2 study English and French. Show these figures on a Venn diagram and hence calculate the probability that:

- **a** a student chosen at random will study none of these languages
- **b** a student chosen at random studies only Spanish
- **c** a student chosen at random studies only English
- **d** a student studies Spanish, given that they study English
- **e** a student does not study Spanish, given that they study English.

9 There are two pre-IB classes, class 1 and class 2. The students in the two classes are asked whether they want to take MAA or MAI when they start the diploma programme.

Maths choice	MAA	MAI
class PIB1	6	12
class PIB2	13	9

Find the probability that a student chosen at random will be:

- **a** a student in class 1
- **b** a student choosing MAA in class 2
- **c** a student choosing MAI
- **d** a student studying MAI given they are in class PIB1
- **e** a student in class PIB2 given they are studying MAA.

Investigation 9.2

When a red die and a blue die are rolled there are 36 possible outcomes which can be shown in a sample space.

Red dice

Blue dice

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

1 Find the probability of rolling:

- **a** a one on the red dice
- **b** a six on the blue dice
- **c** a one on the red dice and a six on the blue dice.

2 Find the probability of rolling:

- **a** an odd number on the blue dice
- **b** a number less than three on the red dice
- **c** an odd number on the blue dice and a number less than three on the red dice.

3 Suggest a relationship, in terms of $P(A)$ and $P(B)$, between the probability of event A and B both occurring.

The events in the question above are **independent** which means the outcome of one does not affect the outcome of the other.

A version of the rule conjectured above will work with events that are not independent, but care needs to be taken to multiply the correct probabilities.

4 In a class of 30 students 16 are boys and 14 are girls. Ten of the girls have at least one brother or sister and so do 11 of the boys.

- **a** Show this information in a contingency or two-way table.
- **b** If a child is chosen at random from the class find the probability that:
  - **i** they are a girl
  - **ii** they have at least one brother or sister given they are a girl
  - **iii** they are a girl **and** have at least one brother or sister
- **c** If a child is chosen at random from the class find the probability that:
  - **i** they are a boy
  - **ii** they have no brother or sister given they are a boy
  - **iii** they are a boy **and** have no brother or sister
- **d** Use your answers to parts b and c to suggest what is the probability of both A and B occurring when A and B are dependent events.

Key point

For independent events, $P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$ and for dependent events $P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B \text{ given event } A)$

9.3 Tree diagrams

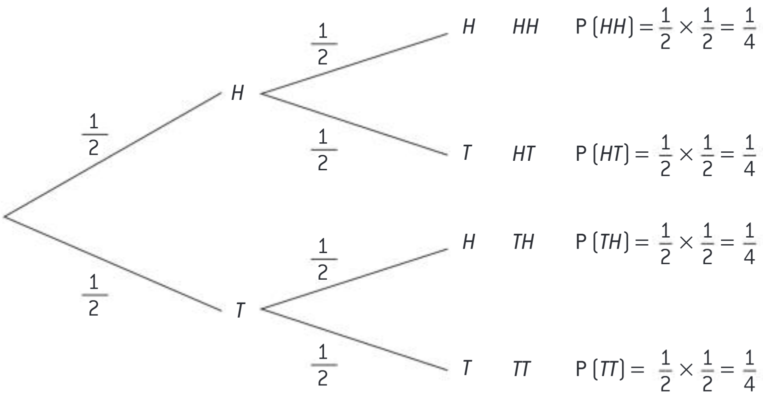
You saw how a sample space diagram can be used to illustrate the possible outcomes when you roll two dice. If you rolled three or more dice then such a diagram would no longer be useful. Another diagram that is used is called a **tree diagram**.

Each time a coin is tossed, the outcome will either be a head or a tail and the probability of each event will be $\frac{1}{2}$.

The event of a head (H) followed by another head is often written as HH.

As we want a head **and** then a second head, we use the fact that for independent events $P(HH) = P(H) \times P(H)$

The tree diagram below shows the four possible outcomes when tossing a coin twice and the associated probabilities.



Key point

A tree diagram shows a sequence of events. Each event has different outcomes shown by individual branches to which you assign probabilities.

An easy way to remember how to work out the probabilities of the combined events is that to find the probability of a particular outcome you multiply the probabilities on the branches leading to that outcome.

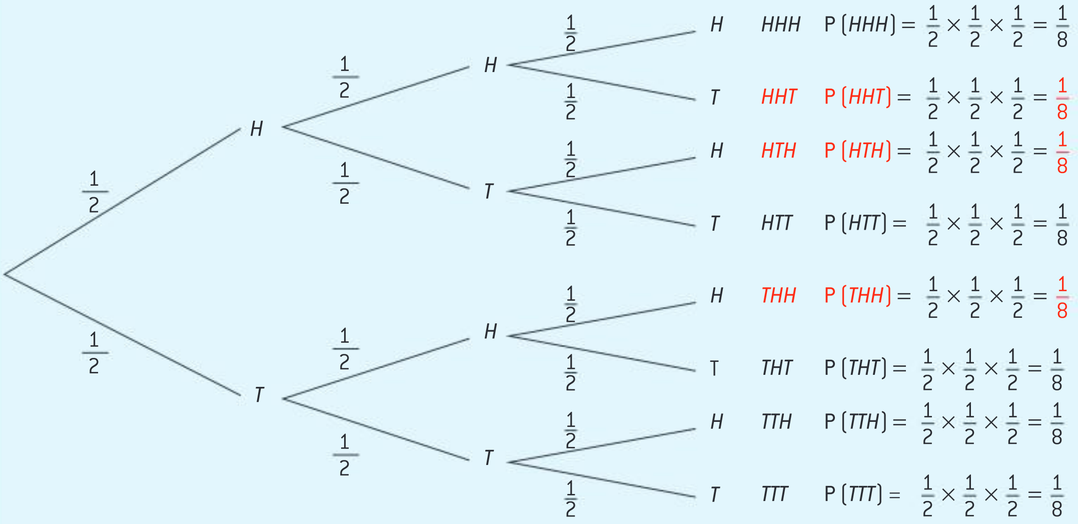
For example, $P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

To combine more than one outcome – for example *HT* **or** *TH* – you add the probabilities: $P(HT \text{ or } TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Notice the events will always be mutually exclusive as you cannot go down two branches at the same time, so we can always add the probabilities.

Notice how, at each stage, the sum of the probabilities is 1 $\frac{1}{2} + \frac{1}{2} = 1$ and $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. This is because at each stage of the tree, the branches are **exhaustive**, that is the events include all possible outcomes.

Example 6

Three coins are tossed. What is the probability of getting 2 heads and 1 tail (in any order).



Extend the tree diagram by adding another set of branches.

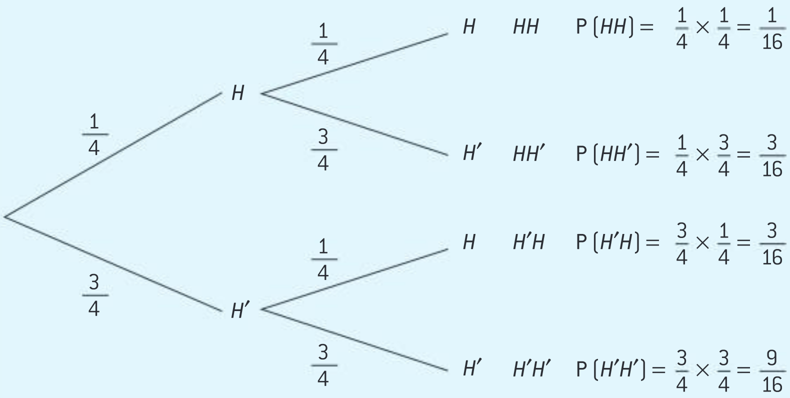
At each stage P(*H*) =½ and P(*T*) = ½.

To get 2 heads and 1 tail, you could get *HHT*, *HTH* or *THH*

$P(HHT, HTH \text{ or } THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Example 7

In a deck of 52 cards, 13 cards are labelled H. A card is drawn and replaced. A second card is drawn. What is the probability that one card is labelled H and the other is not?



If one card is labelled H and the other is not this can be either $(H\,H')$ or $(H'\,H)$.

$$\mathrm{P}(1\ H) = \frac{3}{16} + \frac{3}{16} = \frac{3}{8}$$

13 out of 52 cards are labelled H, hence

$$\mathrm{P}(H) = \frac{1}{4}$$

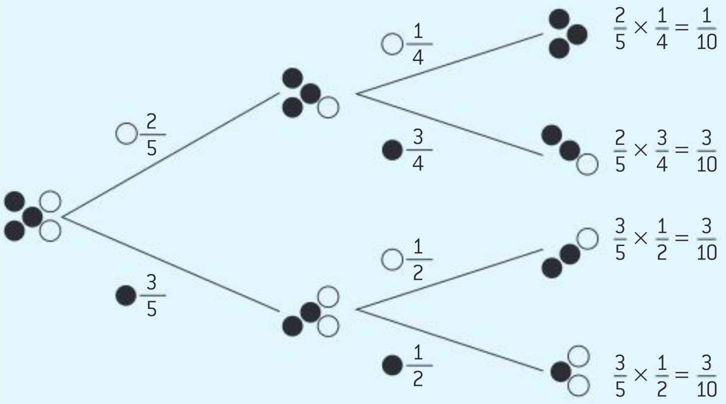
$$\mathrm{P}(H') = 1 - \frac{1}{4} = \frac{3}{4}$$

When the second card is drawn, the probabilities remain the same

Example 8

There are 5 counters; 3 of them are black and 2 are white. After taking two counters, find the probabilities of 3, 2, or 1 black counter remaining. Check that the sum of these probabilities is 1.

What will happen after you take a third counter? Find the probabilities of 2, 1 or 0 black counters remaining.



$$\mathrm{P}(3\text{ black}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

$$\mathrm{P}(2\text{ black}) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{1}{2} = \frac{3}{5}$$

$$\mathrm{P}(1\text{ black}) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

$$\frac{1}{10} + \frac{3}{5} + \frac{3}{10} = 1$$

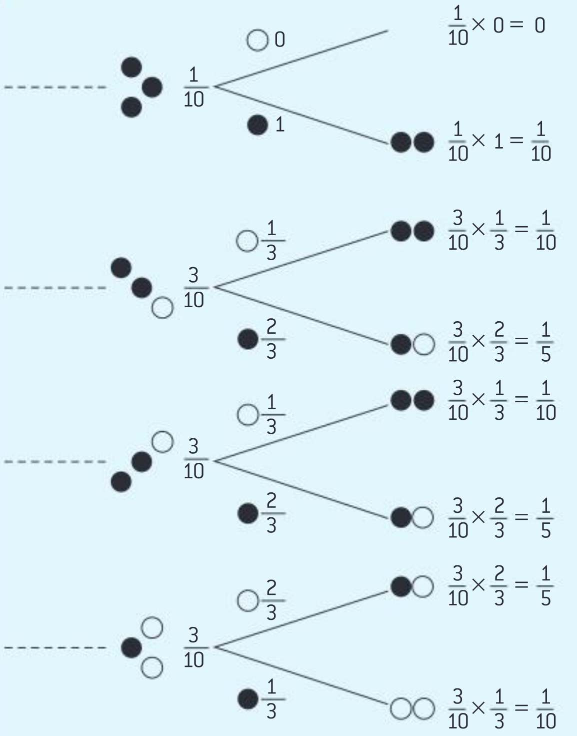
After taking 1 counter, 4 remain. The probability of taking a white counter is $\frac{2}{5}$ and if this happens there will be 3 black counters remaining.

The probability of taking a black counter is $\frac{3}{5}$ and if this happens there will be 2 black counters remaining.

If you go one stage further, you can take either a black or white counter and the probabilities of doing this will vary according to what has been taken the first time.

Calculate the probabilities of each outcome by multiplying along the branches.

There are two ways of getting 2 black counters, so you must add these results together to get the combined probability.



Continue the tree to the next stage and multiply the probabilities along the branches.

$$P(2\text{ black}) = \frac{1}{10} \times 1 + \frac{3}{10} \times \frac{1}{3} + \frac{3}{10} \times \frac{1}{3} = \frac{3}{10}$$

$$P(1\text{ black}) = \frac{3}{10} \times \frac{2}{3} + \frac{3}{10} \times \frac{2}{3} + \frac{3}{10} \times \frac{2}{3} = \frac{3}{5}$$

$$P(0\text{ black}) = \frac{3}{10} \times \frac{1}{3} = \frac{1}{10}$$

Add the probabilities together to get the 3 combined probabilities.

Exercise 9.3

1 The faces of an unbiased dice are painted so that 4 are green and 2 are blue. The dice is rolled twice. Draw a tree diagram to show the possible outcomes and their probabilities. Find the probability that:

- **a** both faces are green
- **b** both faces are the same colour
- **c** both faces are different colours.

2 A fair dice is rolled and a coin is tossed. Draw a tree diagram to show the possible outcomes and the probabilities of this experiment. Find the probability that:

- **a** a 5 is rolled and a tail is tossed
- **b** an even number is rolled and a head is tossed.

3 A coin is biased so that the probability of a head is 0.6 and the probability of a tail is 0.4. The coin is tossed 3 times. Draw a tree diagram to show the outcomes and their probabilities. Calculate the probability of obtaining:

a 3 heads **b** 3 tails **c** 1 head **d** at least 2 heads.

4 A spinner has 5 sectors: 3 red and 2 blue. A second spinner has 4 sectors: 3 red and 1 blue. On each spinner, each sector is equally likely. The two spinners are spun at the same time. Draw a tree diagram to show the outcomes and their probabilities. Find the probability of:

a 2 red sectors **b** 1 red and 1 blue sector **c** 2 blue sectors.

5 A card is taken from a set of 52 cards. 12 of these cards are green. A second card is then taken, without the first being replaced.

Use a tree diagram to find the probabilities that:

a two green cards are drawn **b** one green card is drawn **c** no green cards are drawn.

DP style **Analysis and Approaches HL**

6 Daya and Raquel are playing tennis. The probability Daya wins a game is $\frac{1}{3}$. They play three games.

a Find the probability Daya loses all three.

b Hence write down the probability that Daya wins at least one game.

c Find the probability that both Daya and Raquel win at least one game.

DP style **Applications and Interpretation HL**

7 An environmentalist wishes to test how many fish in a lake are suffering from a disease which is infecting the lake. One of the early indicators of the disease is grey patches appearing on the skin. It is also known that similar patches occur naturally in 10% of the fish population.

The environmentalist catches 80 fish and finds that 30 have the grey patches.

Let p be the probability a fish has the disease. Assume all fish with the disease have the grey patches and the sample caught is representative of the population of the fish in the lake.

a Draw a tree diagram to show the possible ways that a caught fish has grey patches, and write down the corresponding probabilities on your diagram.

b Hence find an estimate for the proportion of fish in the lake with the disease.

Chapter summary

☐ The probability of an event happening $= \dfrac{\text{number of ways it can happen}}{\text{total number of outcomes}}$

☐ Relative frequency of an event $= \dfrac{\text{number of times an event occurs}}{\text{total number of trials}}$

☐ A trial is an individual repetition of an experiment

☐ $\mathrm{P}(A) = \dfrac{n(A)}{n(U)}$

☐ $\mathrm{P}(A') = 1 - \mathrm{P}(A)$

☐ A tree diagram shows a sequence of events. Each event has different outcomes shown by individual branches to which you assign probabilities.

☐ $\mathrm{P}(A \text{ or } B) = \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ whenever A and B are mutually exclusive

☐ For independent events $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ and for dependent events $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \text{given event } A)$

Chapter 9 test

1 A box contains 3 red balls, 2 blue balls and 1 white ball.

Find the probability that a ball taken at random is:

a red **b** blue or white

c yellow **d** not white.

2 A bag contains some coins. There are twice as many 10 ¢ coins as 20 ¢ coins. Find the probability that a coin taken at random from the bag is a 10 ¢ coin.

3 A standard dice is rolled. Find the probability that a number greater than 3 is rolled.

4 Two coins are tossed. Find the probability that the sides uppermost are:

a the same **b** different.

5 Two standard dice are rolled. Find the probability that the product (i.e. the result when they are multiplied) of the two numbers is:

a odd **b** even.

6 The faces of a regular six-sided dice are numbers 1, 1, 1, 3, 6, 6. The dice is rolled twice. Draw a sample space diagram and use it to find the probability of getting a total of:

a 2 **b** 4 **c** 6

d 7 **e** 9 **f** 12

7 A tetrahedral dice has 4 faces numbered 1, 2, 3, 4 and an octahedral dice has eight faces numbered 1, 2, 3, 4, 5, 6, 7, 8.

The two dice are rolled together. Show the outcomes on a sample space diagram and use it to find the probability that the total is:

a greater than 9

b less than 4

c 6 or 7 or 8 or 9.

8 A spinner has eight equal sectors numbered 1 to 8. *A* is the event "the spinner lands on an even number" and *B* is the event "the spinner lands on a number less than 5".

Draw a Venn diagram showing the outcomes and the events *A* and *B*. Use the diagram to find the probability of getting:

a an even number less than 5

b an odd number greater than or equal to 5.

9 In class IB2, there are 16 students. 8 of the students wear glasses and 9 have black hair. There are 4 students who do not wear glasses and who do not have black hair.

Show the figures in a Venn diagram and hence find the probability that a student chosen from the class at random

a wears glasses and has black hair

b has black hair but does not wear glasses

c has black hair given that they wear glasses.

10 In a set of 52 cards, 4 cards are gold. One card is drawn from this set. The card is noted and replaced and another is drawn. Draw a tree diagram and use it to find the probability that both cards are gold cards.

11 The probability of rain on any one day is 0.35. Find the probability of there being no rain and show the different possible outcomes over two successive days on a tree diagram.

Find the probability of:

a rain on two days

b rain on just one of the days

c no rain on either day.

12 Look again at the set of cards used in question 10. A card is drawn from this set. The card is retained and another is drawn. Draw a tree diagram and use it to find the probability that both cards are gold cards. Compare this result to that you got to question 10.

DP style **Applications and Interpretation SL**

13 Paulina and Leah are doing a survey on whether or not students are happy, overall, with the food in the cafeteria.

They record the responses to their survey in the table below:

	Happy	Not happy
Students surveyed by Paulina	20	32
Students surveyed by Leah	12	16

a Use the data to find the approximate proportion of students who are happy with the food in the cafeteria.

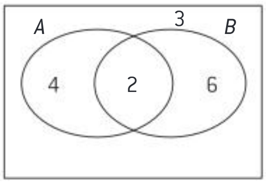
b Vince was surveyed and said he was happy with the food. Find the probability he was surveyed by Leah.

The principal decides to select two students at random and will ask them the same question. If they are both unhappy with the food in the cafeteria he will organize a food committee.

c Find the approximate probability that a food committee with be organised.

DP style **Analysis and Approaches SL**

14 The Venn diagram below shows the number of elements in each region.



Find

a $P(A)$

b $P(B')$

c $P(A \cup B)$

d $P\ (A \cap B)'$

Higher Level

DP style **Analysis and Approaches HL**

15 Students collected weather data for two weeks. They found on 8 days the sun shone and it rained on 10 days. On 3 days there was no rain and no sunshine.

One of the 14 days is selected at random. By drawing a Venn diagram, find the probability that on the day chosen it was sunny with no rain.

DP style **Applications and Interpretation HL**

16 A match in a tennis tournament is played as ‘the best of three sets’. The two players play each two sets and if either player wins both then the game is over. If they have each won one set then the third is played.

Cheryl is playing in this tournament. She knows that her chance of beating her opponent in any set is $\frac{1}{3}$.

a Draw a tree diagram to show the possible outcomes of the match and hence calculate the probability Cheryl will win.

b Cheryl wins the first set. Write down the probability she will go on to win the match.

Modelling and investigation

DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Communication: Reading, writing and using language to gather and communicate information

Organization skills: Managing time and tasks effectively

One of the oldest problems in probability was suggested by Chevalier de Méré, a French nobleman:

The first variation was to roll a dice four times and winning if you get a six.

The second variation involved rolling two dice twenty-four times and winning if you get a double six.

Méré argued that if the chance of getting a six in one roll of the dice was $\frac{1}{6}$ then in four rolls it would be $\frac{4}{6} = \frac{2}{3}$ which is greater than a half. With two dice he figured that if there were 36 possibilities and one of these was a double six then the chance of a double six was $\frac{1}{36}$. With twenty-four rolls, he had a $\frac{24}{36} = \frac{2}{3}$ chance of getting double six, the same as rolling one dice and again more than a half. So Méré deduced that he had more chance of winning than losing.

Can you see anything wrong with Méré's argument? Pascal and Fermat were able to prove him wrong. Here you are going to follow their argument and arrive at the probabilities of both events.

Méré was correct in his initial statement that the chance of getting a six in one roll of the dice is $\frac{1}{6}$.

1 **a** Draw a tree diagram showing the four throws of a dice where the outcomes for each throw are six (S) and not a six (S').

b Write the probabilities of S and S'. (You do not need to write these on every branch of the tree diagram.)

Calculating the probability of getting "at least one six in four throws" (this is equivalent to finding the complementary event of "not getting at least one six in four throws".)

c Rewrite "not getting at least one six in four throws" in terms of S'.

d Use your tree diagram to find this probability of "not getting at least one six in four throws".

e Use the result $P(S') = 1 - P(S)$ to find the probability of getting "at least one six in four throws".

f Write this probability as a decimal to find whether it is greater than or less than a half.

Looking at the two dice problem is obviously more complicated, but the method of calculating the probability is the same.

2 **a** Draw a sample space diagram to find the probability of scoring a double six in one throw. Show that this probability $P(D) = \frac{1}{36}$ and write down $P(D')$

Drawing a tree diagram to show the results of rolling two dice 24 times in not practical. However, to solve the problem, you only need to use one branch of the tree and the pattern of this branch is the same regardless of its length.

b Rewrite the complement of "getting at least one double six in twenty-four rolls" in terms of D'.

c Without drawing the tree diagram, calculate the probability of "not getting at least one double six in twenty-four rolls" and hence of "getting at least one double six in twenty-four rolls".

d Write this probability as a decimal to find whether it is greater than or less than a half.

3 From your results to 1 and 2, explain why Chevalier de Méré's arguments are wrong and why he was consistently losing the game.

10

Speed, distance and time

Learning outcomes

In this chapter you will learn about:

- speed $= \frac{\text{distance}}{\text{time}}$
- Average rate of change of y with respect to x is $\frac{\text{change in } y}{\text{change in } x}$.

Key terms

- Speed
- Gradient
- Distance-time graph
- Rate of change

10.1 Speed, distance and time

Key point

Speed is the rate of change of distance with time.

Key point

speed $= \frac{\text{distance}}{\text{time}}$

Speed is a measurement of how fast something is moving. When an object travels with a constant speed, then its speed, distance and time are connected by the formula: speed $= \frac{\text{distance}}{\text{time}}$

The units of speed are distance per unit of time, so they are metres (the unit of distance) per second (the unit of time), sometimes written m/s or more usually m s^{-1}.

1 m s^{-1} is the speed of something that travels 1 metre in 1 second.

When you are measuring the speed of a car travelling along a road or of a tennis ball being served, the unit that is commonly used is kilometres per hour. This may be written km/h, km h^{-1} or kph.

1 km h^{-1} is the speed of something that travels 1 kilometre in 1 hour.

Note

When an object does not travel with a constant speed, the formula above tells you the average speed over the whole journey. Example 1 explains this further.

Internal link

In section 3.1 you learned that the SI unit for speed was m s^{-1}

Example 1

a A bullet travels 3000 m in 2.4 sec. Calculate its speed in ms^{-1}.

b A car travels 320 km in 4 hours. Find its average speed in km/h.

a

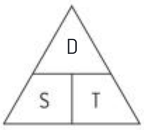
$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{3000}{2.4} = 1250\ \text{ms}^{-1}$$

b

$$\text{average speed} = \frac{\text{distance}}{\text{time}} = \frac{320}{4} = 80\ \text{km/h}$$

The car's speed is not always the same over the journey. Here, we find average speed, which is the total distance travelled divided by the time taken.

You can use a triangle to rearrange the formula speed $= \frac{\text{distance}}{\text{time}}$.

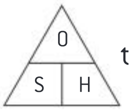


The diagram shows you graphically that:

speed $= \frac{\text{distance}}{\text{time}}$ distance = speed × time time $= \frac{\text{distance}}{\text{speed}}$

Internal link

In Section 6.3 you used a triangle like this



to rearrange the formula sin $= \frac{\text{opp}}{\text{hyp}}$

Example 2

a A cricket ball travels at a speed of 41.1 m/s. The cricket pitch is 20 m long. Calculate how long the ball takes to travel the length of the pitch.

b A plane travels at a speed of 870 km/h. Determine how far it travels in 3½ hours.

a time $= \frac{\text{distance}}{\text{speed}}$

$= \frac{20}{41.1}$

= 0.487 sec (3 s.f.)

b distance = speed × time

= 870 × 3.5

= 3045 km

Questions may involve the conversion of units for speed. For example, since there are 1000 m in 1 km and 3600 sec in 1 hr,

$$1 \text{ km h}^{-1} = 1000 \text{ m h}^{-1}$$
$$= \frac{1000}{3600} \text{ m s}^{-1}$$
$$= 0.278 \text{ m s}^{-1}$$

Internal link

In section 3.1 you learned to convert between different units of measure.

Example 3

A car travels 24 km in 30 min. If average speed is $\frac{\text{distance}}{\text{time}}$, find the car's average speed in:

a km h^{-1}

b m s^{-1}

a average speed is 24 ÷ 0.5 = 48 km h^{-1}

b 24 km = 24 000 m

30 min = 30 × 60 sec = 1800 sec

average speed is 24 000 ÷ 1800 = 13.3 m s^{-1} (3 s.f.)

First convert 30 minutes to 0.5 hours, then use the formula.

Example 4

Calculate the distance travelled by a bus travelling at 85 km h^{-1} for 1 hr 12 min. Give your answer in km.

12 min $= \frac{12}{60} = 0.2$ hours

So total time is 1.2 hours.

distance = speed × time

= 85 × 1.2 = 102 km

First convert 1 hr 12 min to hours.

Note

In Example 4, you could have alternatively converted the time into minutes first. You should try both methods and see what you prefer.

Internal link

You studied finding angles using trigonometry in section 6.3b.

Example 5

a Calculate how many metres a car will travel in 10 seconds at a speed of 50 km h^{-1}.

b Calculate how many minutes the Maglev train takes to travel the distance of 30 km from the airport to Shanghai at an average speed of 257 km/h. Give your answer to the nearest minute.

a $50 \text{ km h}^{-1} = \frac{50 \times 1000}{60 \times 60} \text{ m s}^{-1}$

Convert km h^{-1} to m s^{-1}

distanced travelled $= \frac{50 \times 1000}{60 \times 60} \times 10$

$= 139$ m (3 s.f.)

Use distance = speed × time

b time $= \frac{30}{257}$ h

Use time = distance ÷ speed

$= \frac{30}{257} \times 60$ min

Convert hours to minutes

$= 7$ min

Exercise 10.1a

1 It takes 3½ hours to drive from Sydney to Canberra, a distance of 330 km. Calculate the average speed of the journey.

2 The distance from Amsterdam to New York is 5840 km and the flight takes 8½ hours. Calculate the average speed of the plane on the journey.

3 Calculate the distance travelled by a car travelling at 92 km h^{-1} for 2 hr 20 min

4 Fernanda misses her bus home and walks a distance of 7 km. She can walk at a speed of 5 km/h. Find how long it takes her to walk home (giving your answer in hours and minutes).

5 A glacier has retreated by 750 m in 25 years. Calculate the average speed of retreat of the glacier in centimetres per day during this period.

6 Convert 50 km/h to m/s?

7 A cyclist on a time trial travels 3.5 km in 5 minutes 32 seconds. Calculate her speed in km/h.

DP style **Applications and Interpretation**

8 A river has a width of 40 m and is flowing with a steady current parallel to the banks at a speed of 0.3 m s^{-1}. Philipp wishes to row across the river and can row at a speed of 0.4 m s^{-1}. He begins to row, with his boat pointing perpendicular to the banks.

a Find how long it will take him to reach the opposite bank.

b Find how far the current will have taken him downstream during this time.

c By considering how far he actually travels each second during his crossing, find his actual speed (distance travelled divided by time) relative to the banks.

d Find the angle to the bank at which he is travelling.

A second person leaves at the same time as Philipp from the side of the river directly opposite him, also rowing perpendicular to the banks. They meet when Philipp is 10 m from the bank he is rowing towards.

e Find how far they are downstream from their starting point when they meet.

f Find the speed at which the second person is rowing.

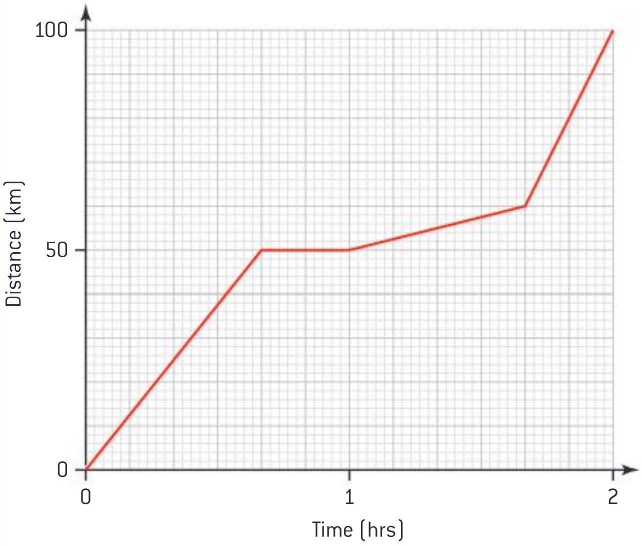
After they have met, Philipp is directly in line with the point he is aiming for on the opposite bank. To ensure he no longer moves downstream he steers the boat at an angle such that if rowing in still water at 0.4 m s^{-1} he would be moving upstream at 0.3 m s^{-1}. This helps to cancel out the effect of the current.

g Use trigonometry to find the angle to the banks at which he should steer so he is no longer moving downstream.

h Find how long it will take him to complete his journey.

Distance and time are often shown on a graph with distance on the vertical axis and time on the horizontal axis. This is known as a **distance-time graph**.

Investigation 10.1a



This graph shows a car that is travelling a distance of 100 km in 2 hours.

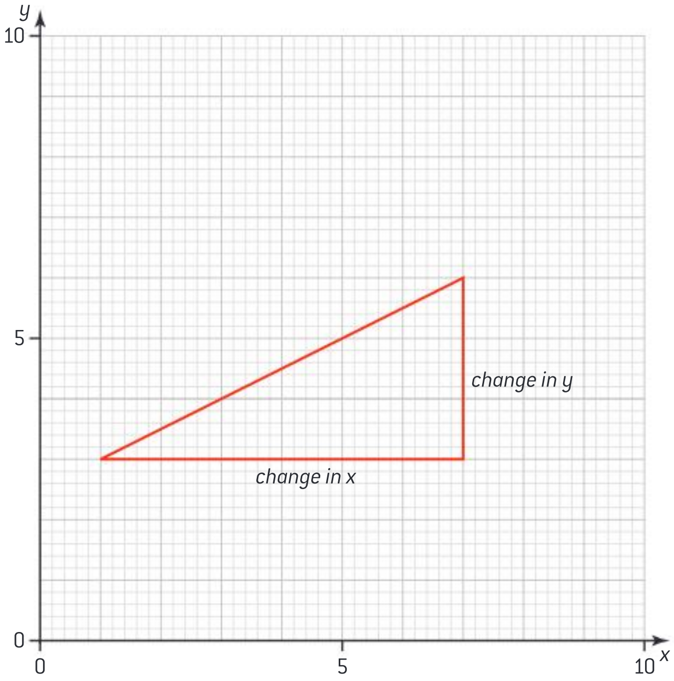
1. How far does the car travel in the first 40 minutes?
2. What is the speed of the car during the first 40 minutes?
3. Describe what is happening between 40 minutes and 1 hour.
4. In which part of the journey is the car moving fastest?
5. What is the speed of the car between $t = 1$ hr and $t = 1$ hr 40 min?
6. What is the speed between $t = 1$ hr 40 min and $t = 2$ hrs?

In a graph, the steeper a line is, the faster the rate of change. A horizontal line means that there is no change. The steepness of a graph is referred to as its gradient.

The horizontal change in a straight line is called the 'change in x' and the vertical change in a straight line is called the 'change in y'.

Internal link

Refer back to section 4.4.



DP link

The gradient function is sometimes written $\frac{dy}{dx}$. You can use this function to find the gradient of a curved line at a certain point on the line. This is part of the topic of calculus and will be studied in detail by all diploma students.

The **gradient** of a straight line is defined as $\frac{\text{change in } y}{\text{change in } x}$

This line has a gradient of $\frac{3}{6} = \frac{1}{2}$

A horizontal line has a gradient of zero and a vertical line's gradient is undefined.

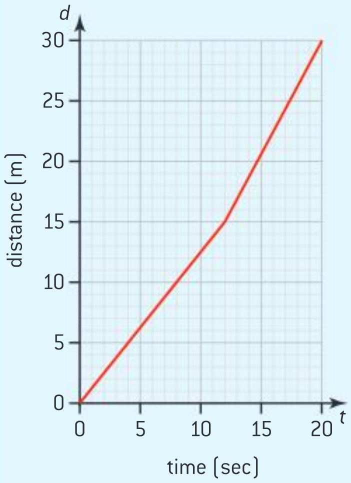
Key point

Speed is the gradient of a distance-time graph

In the distance-time graph, distance travelled is on the vertical axis and the time taken is on the horizontal axis. The change in y is distance travelled and the change in x is the time it takes.

So, the gradient of a distance-time graph is $\frac{\text{distance}}{\text{time}}$ which is speed. As the gradient of a straight line is the same for every point on the line, it follows that the speed is the same for every point on a straight-line distance-time graph.

Example 6



The graph shows the motion of a particle.

a Find the speed of the particle in the first 12 seconds.

b Find the speed of the particle from $t = 12$ to $t = 20$.

c Find the average speed of the particle during the 20 seconds it is in motion.

a $$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{15}{12} = 1.25\ \text{ms}^{-1}$$

The graph is a straight line over the first 12 seconds, which means constant speed.

b $$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{30-15}{20-12} = \frac{15}{8} = 1.875\ \text{ms}^{-1}$$

The graph is a different straight line from 12 to 20 seconds, so this represents a different constant speed.

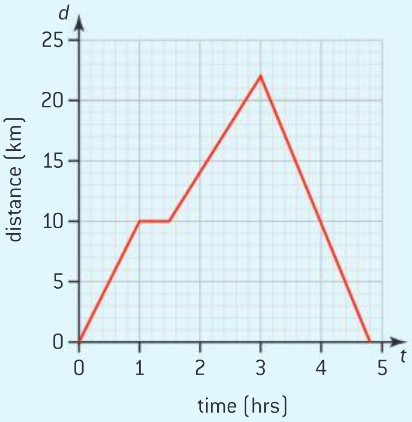
c $$\text{average speed} = \frac{30}{20} = 1.5\ \text{ms}^{-1}$$

Because the speed is not constant over the whole 20 seconds the average speed formula is used.

When an object is travelling so that its distance from a given point is decreasing, it will have a graph with a negative slope. A negative slope is a downward slope and a positive slope is an upward slope.

Example 7

A cyclist sets off from home. His distance from home is shown in this distance-time graph.



a State the time at which the cyclist is stationary.

b Write down the cyclist's furthest distance from home.

c Find the time when the cyclist starts to return home.

d Calculate the cyclist's speed:

- **i** from $t = 0$ to $t = 1$ hr,
- **ii** $t = 1$ hr 30 min to $t = 3$ hrs,
- **iii** $t = 3$ hrs to $t = 4$ hrs 50 min.

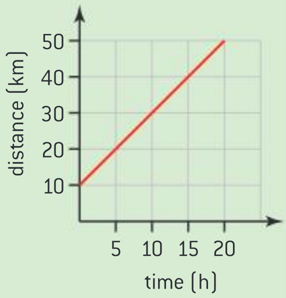
a $t = 1$ hr to $t = 1$ hr 30 min	The graph is horizontal when the cyclist is stationary.
b 22 km	Furthest distance is when the graph changes direction.
c After 3 hrs	
d **i** $\frac{10}{1} = 10 \text{ km h}^{-1}$	
ii $\frac{22-10}{1.5} = \frac{12}{1.5} = 8 \text{ km h}^{-1}$	
iii $\frac{22}{1\frac{5}{6}} = 12 \text{ km h}^{-1}$	1 hr 50 min is $1\frac{5}{6}$ hrs.

Exercise 10.1b

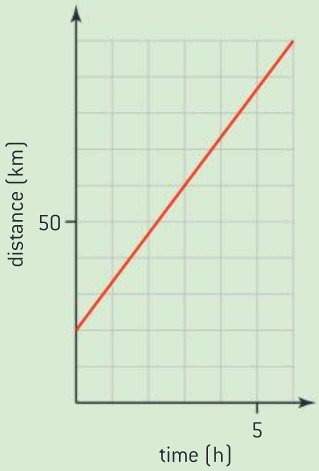
1 Find the speed shown in these distance-time graphs.

a

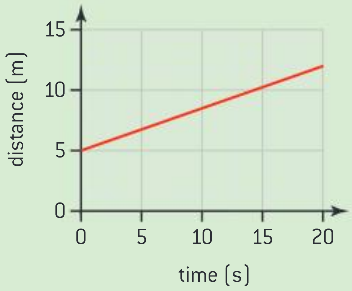
b



c



d



DP link

DP students will study *velocity* which is speed in a given direction. Velocity is a vector quantity and so has direction as well as magnitude. A negative velocity is in the opposite direction to a positive velocity.

Speed is the **magnitude** of the gradient, so the negative sign is ignored.

DP ready Theory of knowledge

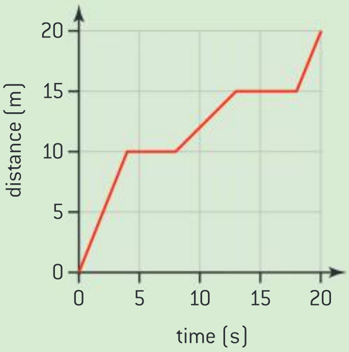
Albert Einstein's theory of special relativity states that the maximum speed at which all matter can travel is the speed of light in a vacuum, 299 792 458 m s^{-1}. Particles in the Large Hadron Collider have reached speeds of over 99% of the speed of light. In the fictional series *Star Trek* and *Star Wars*, space ships are able to travel faster than the speed of light, going into *hyperdrive*.

DP ready **International-mindedness**

Apart from the United States, the United Kingdom and a few island states with close historical ties to the US or UK, all other countries of the world measure speed on the road in km/h. In the US and UK, the units are mph or miles per hour. The mile dates from Roman times and has varied in length throughout history.

For the speed of planes or ships, the most common unit of speed is the knot. A knot is one nautical mile per hour, where a nautical mile is $\frac{1}{60}$ th of a degree of latitude. Because the earth is not an exact sphere, the actual size of $\frac{1}{60}$ th of a degree of latitude varied around the globe. This distance is now defined as exactly 1852 m. Can you use this to calculate how fast 1 knot is when written in km h^{-1}?

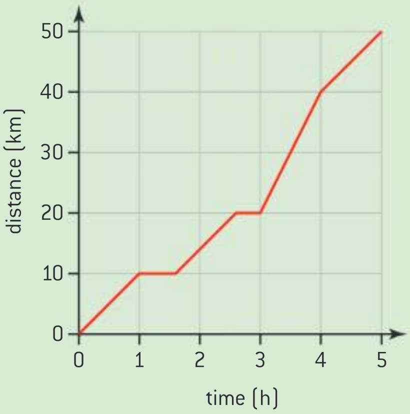
2 The motion of an object that travels 20 m in 20 s is shown in this graph.



Find the speed of the object:

- **a** from $t = 0$ to $t = 4$
- **b** from $t = 4$ to $t = 8$
- **c** from $t = 8$ to $t = 13$
- **d** from $t = 13$ to $t = 18$
- **e** from $t = 18$ to $t = 20$
- **f** Find the average speed from $t = 0$ to $t = 20$.

3 This graph shows the motion of a bicycle.



- **a** Write down the times when the bicycle is stationary.
- **b** Write down the times when the bicycle is travelling at a speed of 10 km/h.
- **c** Calculate the speed between $t = 3$ and $t = 4$.
- **d** Find the average speed of the bicycle for the whole journey.

10.2 Rates of change

A rate of change is a measure of how one variable changes with respect to another.

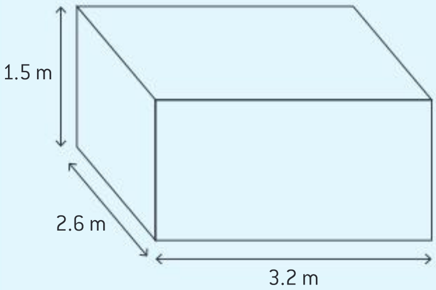
Key point

Average rate of change of y with respect to x is $\frac{\text{change in } y}{\text{change in } x}$.

If time is the second variable (in place of x) then we could just talk about the rate of change of y (or even just the rate of y) and the 'with respect to time' is assumed.

Example 8

Water is pumped into a cuboid shaped tank with height 1.5 m, length 3.2 m and width 2.6 m.



a Find the volume of the tank.

It takes 45 minutes to fill the tank.

b Find the average rate at which water is pumped into the tank. Give your answer in litres per second (one litre is 1000 cm^3).

a $1.5 \times 3.2 \times 2.6 = 12.48\ \text{m}^3$	
b The volume of water in the tank goes from 0 to $12.48\ \text{m}^3$ in 45 minutes	We are looking for the average rate of change of the volume of water in the tank with respect to time.
Change in volume = $\frac{12.48 \times 1\,000\,000}{1000} = 12\,480$ litres	Average rate = $\frac{\text{change in volume}}{\text{time}}$
Time = $45 \times 60 = 2700$ seconds	Convert first to cm^3 and then to litres or directly by knowing that there are 1000 litres in $1\ \text{m}^3$.
Average rate at which the water is added is $\frac{12480}{2700} \approx 4.62$ litres / second	

Exercise 10.2

1 The population of the USA was 282 162 411 in 2000 and 329 093 110 in 2019.

Find the average rate of population increase during this time in number of people per year. Give your answer to 3 significant figures.

2 Acceleration is the rate of change of speed with respect to time. Find the average acceleration of a car that goes from 50 km h^{-1} to 70 km h^{-1} in six seconds.

Give your answers in km h^{-1} s^{-1} (kilometers per hour per second).

3 Water flows into a tank at a rate of 15 gallons per minute.

- **a** Given 1 gallon = 4.55 litres, find the rate at which the water is flowing in litres per second.

  The tank has a volume of 7000 litres.
- **b** Find the length of time it will take to fill the tank. Give your answer in hours and minutes to the nearest minute.

4 The depth of water at a point near the edge of a reservoir is taken at 2 monthly intervals for a year and the values shown in the following graph.

- **a** State the period during which the rate of increase in the height of water in the reservoir is greatest, and find this rate of increase, giving your answer in centimetres per month.
- **b** State the period during which the depth of water is decreasing and find the rate of decrease.
- **c** State by how much the depth of water in the lake has increased over the year.
- **d** Assuming this rate remains constant, estimate the depth of water in January after five more years.

Chapter summary

- ☐ Speed is the rate of change of distance with time
- ☐ $\text{speed} = \frac{\text{distance}}{\text{time}}$
- ☐ When an object does not travel with a constant speed, $\text{speed} = \frac{\text{distance}}{\text{time}}$ tells you the average speed over the whole journey
- ☐ Speed is the gradient of a distance-time graph
- ☐ Average rate of change of y with respect to x is $\frac{\text{change in } y}{\text{change in } x}$. If time is the second variable (in place of x) then we could just talk about the rate of change of y (or even just the rate of y) and the 'with respect to time' is assumed.

Chapter 10 test

1 The average journey time from London to Edinburgh by train is 5 hours 34 minutes. The distance is 534 km. Calculate the average speed for this journey.

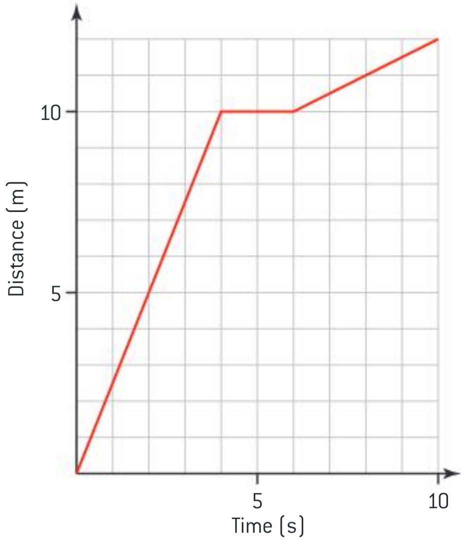
2 Calculate the distance travelled at a speed of 80 km/h in 4 hrs 30 mins.

3 The speed of light is approximately 300 000 km/s. A light-year is the distance light travels in a year. Calculate the distance of a light-year in km. Give your answer in standard form.

4 The Trevi Fountain is 1.5 km from the Roma Termini train station. Find how long it will take to walk there at a speed of 5.1 km/h to the nearest minute.

5 Darragh times how long it takes him to cycle 100 m. If it takes him 23 seconds, find his speed in km/h.

6 The distance travelled by an object is shown in this distance-time graph.



a Find the average speed of the object during these 10 seconds.

b Calculate the speed between $t = 0$ and $t = 4$.

c Find how long the object is at rest for.

d Calculate the speed of the object between $t = 6$ and $t = 10$.

DP style Applications and Interpretation SL

7 The Tokyo-Shin-Aomori train route is 670 km long.

a Given the trains have a top speed of 320 km h^{-1}, find the minimum possible time to complete the journey.

The trains can only travel at top speed for part of the route.

b Given the time to complete the trip is 3 hours measured to the nearest 10 minutes, find the largest possible average speed for the journey.

DP style **Analysis and Approaches SL**

8 Two objects are placed 1.2 m apart at points A and B. At the same time the two objects are projected towards each other.

The object projected from A has a speed of 0.2 m s^{-1} and the object projected from B has a speed of 0.4 m s^{-1}.

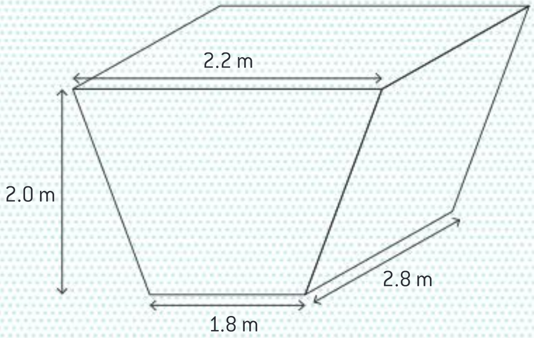
a Write down the distance travelled by each object after *t* seconds.

b Hence find the time at which the objects collide and the distance from A at which the collision occurs.

Higher Level

DP style **Applications and Interpretation HL**

9 An oil tank is in the shape of a prism with a trapezoidal cross-section and with dimensions as shown in the diagram below.



Oil is taken from the tank at a rate of 50 litres per day. Initially the tank is filled with oil and when the depth of oil in the tank drops to 50 cm, more oil is ordered.

Find the length of time until more oil needs to be ordered.

Remember than $1 m^3 = 1000$ litres.

DP style **Analysis and Approaches HL**

10 A particle travels for t_1 seconds at a speed of v_1 m s^{-1}. During this time it covers 10 m. Its speed then changes to v_2 m s^{1} and it travels at this speed for t_2 seconds.

The particle travels for a total of 2 seconds and its average speed during this time is 8 m s^{-1}.

Given v_2 is three times v_1, find the values of v_1, v_2, t_1 and t_2.

Modelling and investigation

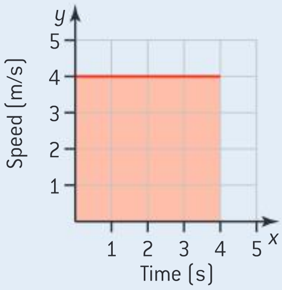
DP ready **Approaches to learning**

Critical thinking: Analysing and evaluating issues and ideas

Organization skills: Managing time and tasks effectively

You can also represent speed and time with a graph. In a **speed-time graph**, speed is shown on the vertical axis and time on the horizontal axis.

In a speed-time graph, a horizontal line shows constant speed.



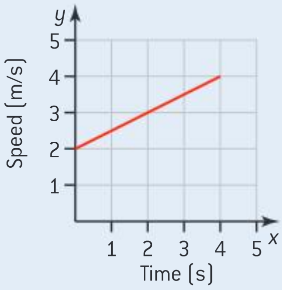
This graph shows an object that is travelling at 4 m/s for 4 seconds.

During its motion distance = speed × time.

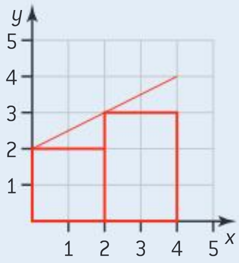
The distance travelled is 4 × 4 = 16 m, which is the area of the rectangle under the line.

If the speed is changing during motion, then the graph will not be horizontal.

This graph shows an object with a speed that increases from 2 m/s to 4 m/s in 4 sec.



We can approximate the motion of the object by assuming it travelled for 2 seconds at 2 m s^{-1} and for 2 seconds at 3 m s^{-1}, as shown in the diagram.

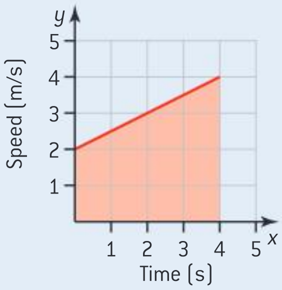


1 **a** Use this approximation to find an estimate for the distance travelled during the object's motion.

b State whether your estimate will be an underestimate or an overestimate.

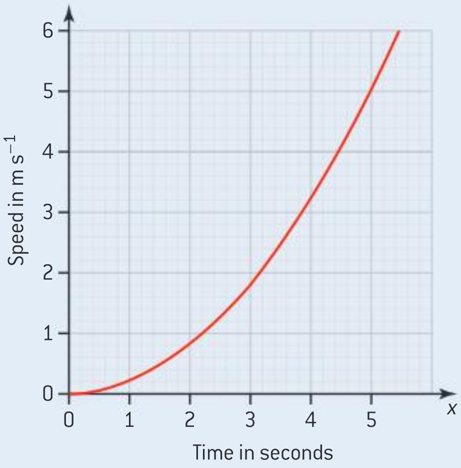
c Give two ways the approximation could be improved.

As hinted above, the smaller the rectangles the more accurate the measure of distance travelled and the closer the result comes to the area between the line and the horizontal axis which leads to the result that the area under a speed-time graph is equal to the distance travelled.



2 Use this result to find the actual distance travelled by the object during the 4 seconds.

The following graph shows the speed of an object in m s^{-1}.



3 **a** Use five rectangles to find an estimate for the distance it travelled in the first 5 seconds.

b State whether your answer will be an underestimate or an overestimate.

c Give one way your estimate could be improved.

4 Try to improve your estimate to obtain an answer that is correct to 1 decimal place. Once you feel you have achieved this, compare your result to the answer given in the solutions to see how close you were.

DP link

You will learn how to find the exact area under this curve when you do the calculus section of the DP course.

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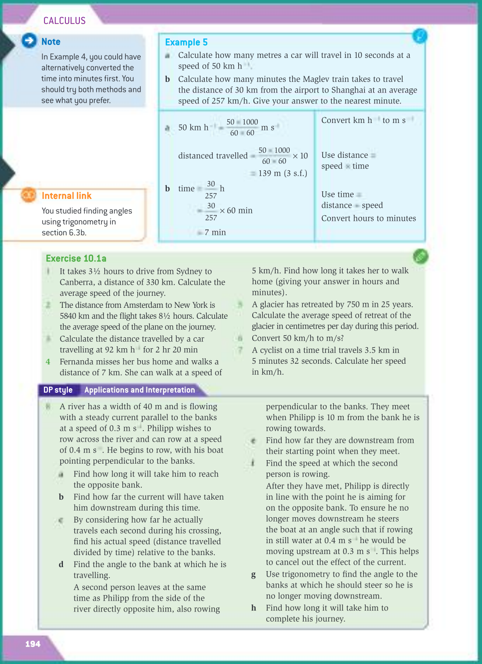
Author

Jim Fensom

Clear explanations and worked examples consolidate understanding

Introduce and explore **Analysis and approaches** and **Applications and interpretation**



CALCULUS

Note

In Example 4, you could have alternatively converted the time into minutes first. You should try both methods and see what you prefer.

Example 5

a Calculate how many metres a car will travel in 10 seconds at a speed of 50 km h^{-1}.

b Calculate how many minutes the Maglev train takes to travel the distance of 30 km from the airport to Shanghai at an average speed of 257 km/h. Give your answer to the nearest minute.

a $50 \text{ km h}^{-1} = \frac{50 \times 1000}{60 \times 60} \text{ m s}^{-1}$ Convert km h^{-1} to m s^{-1}

distanced travelled $= \frac{50 \times 1000}{60 \times 60} \times 10$ $= 139 \text{ m (3 s.f.)}$ Use distance = speed × time

b time $= \frac{30}{257}$ h $= \frac{30}{257} \times 60$ min $= 7$ min Use time = distance ÷ speed Convert hours to minutes

Internal link

You studied finding angles using trigonometry in section 6.3b.

Exercise 10.1a

1 It takes 3½ hours to drive from Sydney to Canberra, a distance of 330 km. Calculate the average speed of the journey.

2 The distance from Amsterdam to New York is 5840 km and the flight takes 8½ hours. Calculate the average speed of the plane on the journey.

3 Calculate the distance travelled by a car travelling at 92 km h^{-1} for 2 hr 20 min

4 Fernanda misses her bus home and walks a distance of 7 km. She can walk at a speed of 5 km/h. Find how long it takes her to walk home (giving your answer in hours and minutes).

5 A glacier has retreated by 750 m in 25 years. Calculate the average speed of retreat of the glacier in centimetres per day during this period.

6 Convert 50 km/h to m/s?

7 A cyclist on a time trial travels 3.5 km in 5 minutes 32 seconds. Calculate her speed in km/h.

DP style **Applications and Interpretation**

8 A river has a width of 40 m and is flowing with a steady current parallel to the banks at a speed of 0.3 m s^{-1}. Philipp wishes to row across the river and can row at a speed of 0.4 m s^{-1}. He begins to row, with his boat pointing perpendicular to the banks.

a Find how long it will take him to reach the opposite bank.

b Find how far the current will have taken him downstream during this time.

c By considering how far he actually travels each second during his crossing, find his actual speed (distance travelled divided by time) relative to the banks.

d Find the angle to the bank at which he is travelling.

A second person leaves at the same time as Philipp from the side of the river directly opposite him, also rowing perpendicular to the banks. They meet when Philipp is 10 m from the bank he is rowing towards.

e Find how far they are downstream from their starting point when they meet.

f Find the speed at which the second person is rowing.

After they have met, Philipp is directly in line with the point he is aiming for on the opposite bank. To ensure he no longer moves downstream he steers the boat at an angle such that if rowing in still water at 0.4 m s^{-1} he would be moving upstream at 0.3 m s^{-1}. This helps to cancel out the effect of the current.

g Use trigonometry to find the angle to the banks at which he should steer so he is no longer moving downstream.

h Find how long it will take him to complete his journey.

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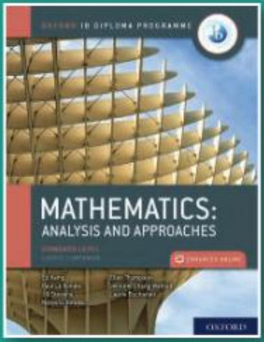

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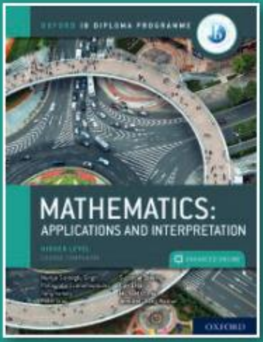

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